2 The Linear Quadratic Regulator (LQR)

Problem:
Compute a state feedback controller

\[ u(t) = Kx(t) \]

that stabilizes the closed loop system and minimizes

\[ J := \int_{0}^{\infty} x(t)^T Q x(t) + u(t)^T R u(t) \, dt \]

where \( x \) and \( u \) are the state and control of the LTI system

\[ \dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0. \]

Assumptions:
a) \( Q \succeq 0, \ R > 0; \)
b) \((A, B)\) stabilizable;

A first step toward a solution:
The closed loop cost is

\[ J = \int_{0}^{\infty} x(t)^T (Q + K^T R K) x(t) \, dt \]

and the closed loop system is

\[ \dot{x} = (A + BK)x, \quad x(0) = x_0. \]

But for a given \( K \) and \( x_0 \)

\[ x(t) = e^{(A+BK)t} x_0. \]

Hence

\[ J = \int_{0}^{\infty} x_0^T e^{(A+BK)t}(Q + K^T R K)e^{(A+BK)t} x_0 \, dt \]

\[ = x_0^T \left( \int_{0}^{\infty} e^{(A+BK)t}(Q + K^T R K)e^{(A+BK)t} \, dt \right) x_0. \]
This means that $J$ can be computed as

$$J = x_0^T X x_0$$

where $X$ is the solution to the Lyapunov equation

$$(A + BK)^T X + X(A + BK) + Q + K^T R K = 0.$$ 

Before proceeding we need to learn how to solve the above Lyapunov equation in $X$ and $K$. This is not always possible. In this case, because $R \succ 0$, we can complete the squares, rewriting the above equation in the form

$$A^T X + X A - X B R^{-1} B^T X + Q + (X B R^{-1} + K^T) R (R^{-1} B^T X + K) = 0.$$ 

Note that $K$ is confined to the term

$$(X B R^{-1} + K^T) R (R^{-1} B^T X + K) \succeq 0$$

and that for

$$K = -R^{-1} B^T X.$$ 

we have

$$Q + (X B R^{-1} + K^T) R (R^{-1} B^T X + K) = Q.$$ 

This reduces the above equation to

$$A^T X + X A - X B R^{-1} B^T X + Q = 0.$$ 

This is an Algebraic Riccati Equation (ARE) in $X$.

As we learn more about AREs we shall prove that the above choice of $K$ and $X$ is so that

a) $A + BK$ is Hurwitz (asymptotically stable);

b) $X$ is “minimum” in a certain sense;

c) The associated $J$ is minimized.
2.1 Comparison Lemma

If $S \succeq 0$ and $Q_2 \succeq Q_1 \succeq 0$ then $X_1$ and $X_2$, solutions to the Riccati equations

$$A^T X_1 + X_1 A - X_1 S X_1 + Q_1 = 0,$$
$$A^T X_2 + X_2 A - X_2 S X_2 + Q_2 = 0,$$

are such that

$$X_2 \succeq X_1$$

if $A - S X_2$ is asymptotically stable.

**Proof:** Note that

$$A^T X_1 + X_1 A - X_1 S X_1 + Q_1 = (A - S X_2)^T X_1 + X_1 (A - S X_2) + X_2 S X_2 + Q_1 - (X_1 - X_2) S (X_1 - X_2),$$

and

$$A^T X_2 + X_2 A - X_2 S X_2 + Q_2 = (A - S X_2)^T X_2 + X_2 (A - S X_2) + X_2 S X_2 + Q_2$$

Now subtract the above equations to obtain the Lyapunov equation

$$(A - S X_2)^T \bar{X} + \bar{X} (A - S X_2) + \bar{Q} = 0$$

where

$$\bar{X} := X_2 - X_1, \quad \bar{Q} := (Q_2 - Q_1) + (X_1 - X_2) S (X_1 - X_2) \succeq 0.$$ 

Therefore, if $A - S X_2$ is Hurwitz we conclude that $\bar{X} = X_2 - X_1 \succeq 0$, that is $X_2 \succeq X_1$. 


We can now use the comparison lemma to compare the two AREs
\[ A^T X_2 + X_2 A - X_2 BR^{-1} B^T X_2 + Q_2 = 0 \]
and
\[ A^T X_1 + X_1 A - X_1 BR^{-1} B^T X_1 + Q_1 = 0 \]
where
\[ S = BR^{-1} B^T \succeq 0, \]
and
\[ Q_1 = Q, \quad Q_2 = Q + (X_2 BR^{-1} + K^T) R (R^{-1} B^T X_2 + K). \]
Note that for any \( X_2 \) and stabilizing \( K \) that
\[ Q_2 = Q + (X_2 BR^{-1} + K^T) R (R^{-1} B^T X_2 + K) \succeq Q = Q_1 \]
because \( R \succ 0 \). Therefore, for any choice of
\[ K \neq -R^{-1} B^T X_1 \]
we shall have
\[ X_2 \succeq X_1. \]
This proves that \( X_1 \) is “minimum”. Of course this also implies that
\[ J_2 = x_0^T X_2 x_0 \geq x_0^T X_1 x_0 = J_1 \]
so that \( J \) is also being minimized.
2.2 More on AREs

**Warning:** In this section we consider Riccati equations of the form

\[ A^T X + XA + XZX + Q = 0 \]

**Lemma 1:** Consider the *Hamiltonian matrix*

\[ H := \begin{bmatrix} A & Z \\ -Q & -A^T \end{bmatrix}. \]

where \( A, Z = Z^T \) and \( Q = Q^T \in \mathbb{R}^{n \times n} \).

1. \( \lambda \) is an eigenvalue of \( H \) if and only if \( -\lambda \) is an eigenvalue of \( H \).

2. If \( H \) has no eigenvalues on the imaginary axis then there exists a matrix \( W \in \mathbb{R}^{n \times n} \) such that

\[ HV_1 = V_1 W \quad (2) \]

where \( W \) is Hurwitz.

Proof:

Item 1. \( H \) has eigenvalues pairs which are symmetric w.r.t the imaginary axis because

\[ J^{-1}HJ = -JHJ = -H^T, \quad J := \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}, \quad J^{-1} = -J \]

Item 2. Let \( H_J \) be the Jordan form of matrix \( H \) so that

\[ HV = VH_J \]

where \( V \in \mathbb{R}^{2n \times 2n} \) is a matrix whose columns are the (generalized) eigenvectors of \( H \). Since the eigenvalues of \( H \) are symmetric with respect to the imaginary axis and there are no eigenvalues on the imaginary axis, there exists at least two distinct Jordan blocks

\[ H \begin{bmatrix} V_1 & V_2 \end{bmatrix} = \begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} H_{J_-} & 0 \\ 0 & H_{J_+} \end{bmatrix} \]

where all \( n \) eigenvalues of \( H_{J_-} \) have negative real part, i.e., \( H_{J_-} \) is Hurwitz. The first columns of the above equation are in the form \( (2) \) with \( W = H_{J_-} \) Hurwitz.
Lemma 2: Consider the Algebraic Riccati Equation (ARE)

\[ A^T X + X A + X Z X + Q = 0 \]

where \( A, Z = Z^T \) and \( Q = Q^T \in \mathbb{R}^{n \times n} \) and the associated Hamiltonian matrix

\[
H := \begin{bmatrix}
A & Z \\
-Q & -A^T
\end{bmatrix}.
\]

which is assumed to have no eigenvalue on the imaginary axis.

1. Let

\[
V_1 = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \in \mathbb{C}^{2n \times n}
\]

be (generalized) eigenvectors of \( H \) associated with all \( n \) eigenvalues with negative real part. If \( X_1 \) is nonsingular then \( X = X_2 X_1^{-1} \) solves the ARE.

2. The solution obtained in item 1. is

(a) real,
(b) symmetric,
(c) unique stabilizing \((A + Z X) \) is Hurwitz).

3. If \( Z \succeq 0 \) (or \( Z \preceq 0 \)) then \( X_1 \) is invertible if and only if \((A, Z)\) is stabilizable.

Proof:
Item 1. From Item 2. of Lemma 1 there exists a Hurwitz matrix \( W \) such that

\[
HV_1 = V_1 W
\]

Then, multiplying the above by \( X_1^{-1} \) on the right and by \([X -I]\) on the left we get

\[
[X -I] H \begin{bmatrix} I \\ X \end{bmatrix} = [X -I] \begin{bmatrix} I \\ X \end{bmatrix} X_1 W X_1^{-1} = 0
\]

Note that

\[
[X -I] H \begin{bmatrix} I \\ X \end{bmatrix} = [X -I] \begin{bmatrix} A & Z \\
-Q & -A^T
\end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix} = A^T X + X A + X Z X + Q
\]
Item 2. (a) Since $H$ is real, the columns of $V_1$ can be chosen complex conjugates in pairs so that

$$\bar{V}_1 = \begin{bmatrix} \bar{X}_1 \\ \bar{X}_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} P = \begin{bmatrix} X_1 P \\ X_2 P \end{bmatrix}$$

where $P$ is some permutation matrix and

$$\bar{X} = \bar{X}_2 \bar{X}_1^{-1} = X_2 P P^{-1} X_1^{-1} = X_2 X_1^{-1} = X$$

that is $X$ is real.

Item 2. (b) $X$ is symmetric if

$$X = X_2 X_1^{-1} = (X_1^{-1})^* X_2^* = X^*$$

or in other words

$$T = X_2^* X_1 - X_1^* X_2 = 0.$$ 

Now note that

$$T = V_1^* J V_1 = \begin{bmatrix} X_1^* & X_2^* \end{bmatrix} \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

Since $J H J = H^T$, $H$ and $J$ satisfy the Lyapunov equation

$$J H + H^T J = 0$$

Multiplying the above equation by $V_1^*$ on the left and $V_1$ on the right and using (2)

$$0 = V_1^* J H V_1 + V_1^* H^T J V_1$$

$$= V_1^* J V_1 W + W^* V_1^* J V_1$$

$$= T W + W^* T.$$ 

Because $W$ is Hurwitz $T = 0$.  

---

MAE 280 B

29

Maurício de Oliveira
Item 2. (c) Multiply (2) by $X_1^{-1}$ on the right and by $[I \ 0]$ on the left to obtain

$$[I \ 0] H \begin{bmatrix} I \\ X \end{bmatrix} = [I \ 0] \begin{bmatrix} A & Z \\ -Q & A^T \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix} = A + ZX = X_1WX_1^{-1}.$$ 

Therefore, $A + ZX$ is stable because it is similar to a stable matrix. For uniqueness assume $\tilde{X}$ is also a stabilizing solution to the ARE. Therefore subtracting the two AREs

$$0 = (A^T X + X A + X ZX + Q) - (A^T \tilde{X} + \tilde{X} A + \tilde{X} Z \tilde{X} + Q)$$

$$= A^T (X - \tilde{X}) + (X - \tilde{X}) A + X ZX - \tilde{X} Z \tilde{X}$$

$$= A^T (X - \tilde{X}) + (X - \tilde{X}) A + X ZX - \tilde{X} Z \tilde{X} - XZ \tilde{X} + XZ \tilde{X}$$

$$= A^T (X - \tilde{X}) + (X - \tilde{X}) A + XZ (X - \tilde{X}) + (X - \tilde{X}) Z \tilde{X}$$

$$= (A + ZX)^T (X - \tilde{X}) + (X - \tilde{X}) (A + Z \tilde{X})$$

The last equation can be seen as a Sylvester equation in $(X - \tilde{X})$ and since $A + ZX$ and $A + Z \tilde{X}$ are both Hurwitz $\lambda_i(A + ZX) + \lambda_j(A + Z \tilde{X}) < 0$ so that it admits only the trivial solution, that is, $X - \tilde{X} = 0$.

Item 3. To prove sufficiency note that if $X_1$ is invertible then $A + ZX$ is stable such that $(A, Z)$ is stabilizable.

The proof of necessity is more complicated. Assume that $Z \succeq 0$ (or $Z \preceq 0$), $(A, Z)$ is stabilizable and that $X_1$ is singular, such that there exists $x \neq 0$ such that $X_1 x = 0$. Multiply (2) by $[I \ 0]$ on the left to obtain

$$[I \ 0] H \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = AX_1 + ZX_2 = X_1 W$$

Multiply on the left by $x^* X_2^*$ and on the right by $x$

$$x^* X_2^* A X_1 x + x^* X_2^* ZX_2 x = x^* X_2^* X_1 W x = x^* X_2^* X_1^* X_2 W x$$

and use the fact that $X_1 x = 0$ to obtain

$$x^* X_2^* ZX_2 x = 0$$

which implies $ZX_2 x = 0$ because $Z \succeq 0$ (or $Z \preceq 0$). Note that this also implies

$$X_1 W x = 0.$$
**Auxiliary lemma:** Assume $W$ is Hurwitz. There exists $x \neq 0$ such that $X_1x = X_1Wx = 0$ if and only if there exists $\tilde{x} \neq 0$ such that

$$X_1\tilde{x} = 0, \quad W\tilde{x} = \tilde{\lambda}\tilde{x}, \quad \tilde{\lambda} + \tilde{\lambda}^* < 0.$$  

Proof (Auxiliary lemma): Sufficiency is immediate since

$$X_1W\tilde{x} = \tilde{\lambda}X_1\tilde{x} = 0.$$  

Necessity follows by contradiction. If $X_1$ is singular and

$$\not\exists \tilde{x} : X_1\tilde{x} = 0, \quad W\tilde{x} = \tilde{\lambda}\tilde{x}, \quad \tilde{\lambda} + \tilde{\lambda}^* < 0$$

then

$$X_1W\tilde{x} = \tilde{\lambda}X_1\tilde{x} \neq 0$$

for any eigenvalue/eigenvector pair $(\tilde{\lambda}, \tilde{x})$. Because $W$ is nonsingular this must be true for $n$ linearly independent vectors, which implies that $X_1$ is not singular. \hfill \Box (Auxiliary lemma)

Now multiply (2) by $[0 \ I]$ on the left to obtain

$$[0 \ I] H \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = -QX_1 - A^*X_2 = X_2W$$

and multiply by $\tilde{x}$ such that $X_1\tilde{x} = 0$ on the right

$$0 = (A^*X_2 + X_2W)\tilde{x}$$

$$= (A^* - \lambda I)X_2\tilde{x}, \quad \lambda = -\tilde{\lambda}, \quad \lambda + \lambda^* > 0.$$  

Since $ZX_2\tilde{x} = 0$, this implies that $(A, Z)$ is not stabilizable, which is a contradiction.
We now use Lemma 2 to prove that \( A + BK \) is Hurwitz. It amounts to apply item 3 of Lemma 2 to the ARE

\[
A^T X + X A - X B R^{-1} B^T X + Q = 0.
\]

For that notice that

\[
Z = -B R^{-1} B^T \preceq 0
\]

because \( R \succ 0 \) then \( B R^{-1} B^T \succeq 0 \). Therefore, if \( (A, Z) = (A, -B R^{-1} B^T) \) is stabilizable then \( X_1 \) in Lemma 2 should be invertible and the solution \( X \) should be unique, symmetric and stabilizing.

With that in mind suppose that \( (A, B) \) is stabilizable but that \( (A, -B R^{-1} B^T) \) is not, so that there exists \( z \neq 0 \) such that

\[
z^* A = \lambda z^*, \quad z^* B R^{-1} B^T = 0, \quad \lambda + \lambda^* \geq 0.
\]

Therefore

\[
z^* B R^{-1} B^T z = 0.
\]

Because \( R^{-1} \succ 0 \) this can only be true if \( z^* B = 0 \), which contradicts the hypothesis that \( (A, B) \) is stabilizable, proving that \( (A, -B R^{-1} B^T) \) is stabilizable and

\[
A + BK = A + B(-R^{-1} B^T X) = A - BR^{-1} B^T X = A + ZX
\]

is Hurwitz.

**Warning:** The optimal control gain \( K \) is independent from the initial condition! However, the optimal cost is not!