The Bounds on the Coefficients of Restitution for the Frictional Impact of Rigid Pendulum Against a Fixed Surface

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Upper bounds on Newton's, Poisson's, and energetic coefficients of normal restitution for the frictional impact of rigid pendulum against a fixed surface are derived, demonstrating that the upper bound on Newton's coefficient is smaller than 1, while the upper bound on Poisson's coefficient is greater than 1. The upper bound on the energetic coefficient of restitution, which is a geometric mean of Newton's and Poisson's coefficients of normal restitution, is equal to 1. Lower bound on all three coefficients is equal to zero. The bounds on the tangential impact coefficient, defined by the ratio of the frictional and normal impulses, are also derived. Its lower bound is negative, while its upper bound is equal to the kinetic coefficient of friction. Simplified bounds in the case of a nearly vertical impact are also derived. [DOI: 10.1115/1.3172198]

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1 Introduction

The determination of the rebounding velocity components of colliding bodies is an old mechanics problem, with its origin in early work by Newton and Poisson. Newton defined the coefficient of normal restitution as the ratio of the relative normal velocities after and before the impact. In contrast to Newton's kinematic definition, Poisson's kinetic definition is based on the ratio of the magnitudes of the normal impulses corresponding to the periods of restitution and compression. In the absence of friction (frictionless impact), the Poisson definition of the coefficient of normal restitution yields the same expression, in terms of the relative velocities, as does the Newton definition, which is demonstrated in standard dynamics textbooks, e.g., Ref. [1]. In the presence of friction, however, the two definitions are, in general, not equivalent. The simplest theory of the frictional impact is that of Whittaker [2], in which it is assumed that the frictional impulse is in the slip direction and is equal to the product of the coefficient of friction and the magnitude of the normal impulse. Kane [3] observed that this theory leads to an increase in kinetic energy upon the impact of a double pendulum with a rough horizontal surface, for some values of the coefficients of friction and normal restitution, and for some kinematic parameters of motion. Keller [4] explained this by noting that Whittaker's theory [2] applies only when the direction of sliding is constant throughout the collision. If there is a reversal of the slip direction during the impact process, the coefficient of the proportionality between the tangential and normal impulses is different from the coefficient of kinetic friction. Keller's [4] analysis also demonstrated the advantage of using the normal impulse as an independent variable, instead of physical time, to cast and analyze the governing differential equations of motion during the impact process. Stronge [5] introduced an energetic coefficient of normal restitution, whose square is equal to the negative ratio of the elastic strain energy released during restitution and the internal energy of deformation absorbed during compression phase of the impact. Numerous papers, proposing different models of frictional impact, were published since. A comprehensive treatment of the subject, with a historical outline, can be found in the monographs or review articles [6–9].

In this paper we revisit a classical problem of frictional impact of rigid pendulum against a fixed surface. By employing Keller's [4] method of analysis, we derive the expressions for the angular velocity in terms of the monotonically increasing normal impulse during the impact process. Three different definitions of the coefficient of normal restitution are used to specify the rebounding angular velocity and the total normal impulse: Newton's kinematic, Poisson's kinetic, and Stronge's energetic definitions. It is shown that the energetic coefficient of normal restitution is a geometric mean of the Newton and Poisson coefficients of normal restitution. The upper bounds on all three coefficients are established, demonstrating that the upper bound on the Newton coefficient is smaller than 1, while the upper bound on the Poisson coefficient is greater than 1. For the pendulum striking a rough surface elastically, without dissipation due to deformation, the Newton and Poisson coefficients are the reciprocals of each other. If, upon the impact, the pendulum sticks to the ground, there is no restitution phase of the impact, and all three coefficients of normal restitution are equal to zero, which represents their lower bound. The bounds on the tangential impact coefficient, defined by the ratio of the frictional and normal impulses, are also derived. Its lower bound is negative, while its upper bound is equal to the kinetic coefficient of friction. Simplified bounds in the case of a nearly vertical impact are also derived.

2 Rigid Pendulum Striking a Fixed Surface

Figure 1 shows a rigid pendulum, rotating around a frictionless pin at O and striking a fixed horizontal surface at the point with coordinates $(a,-b)$, relative to the origin at O. The (incidence) angular velocity of the pendulum just before the impact is $\omega^- < 0$ (negative value indicating its clockwise direction). If $\omega^+ > 0$ is the (rebounding) angular velocity immediately after the impact of duration t_1, then, by the impulse principle,

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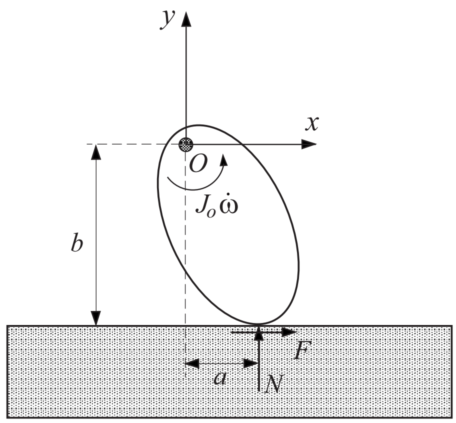

Fig. 1 A rigid pendulum, suspended from a frictionless pin *O*, strikes a fixed horizontal surface with the incidence angular velocity ω^-. The componential reactions at the contact point with the coordinates $(a,-b)$ are *N* and *F*.

$$J_0\omega^- + \int_0^{t_1} (Na + Fb)\mathrm{d}t = J_0\omega^+ \tag{1}$$

where J_0 is the pendulum's moment of inertia about the point O, and N and F are the normal and friction forces acting on the pendulum at the contact point with the rough horizontal surface. The coordinates of the contact points a and b change only infinitesimally during the time of the impact $t \in [0, t_1]$, so that the equation of motion during the impact is

$$J_0 \frac{\mathrm{d}\omega}{\mathrm{d}t} = Na + Fb \tag{2}$$

The weight of the pendulum mg, as a nonimpulsive force, does not contribute to Eqs. (1) and (2). Following Ref. [4] and introducing a monotonically increasing impulse parameter (Fig. 2)

$$\tau = \int_0^t N\mathrm{d}t, \quad \mathrm{d}\tau = N\mathrm{d}t \tag{3}$$

Eq. (2) can be recast as

$$J_0 \mathrm{d}\omega = \left(a + b\frac{F}{N}\right)\mathrm{d}\tau, \quad \tau \in [0, \tau_1], \quad \tau_1 = \int_0^{t_1} N\mathrm{d}t \tag{4}$$

Since a and b are nearly constant during the impact process, Eq. (4) can be integrated to give

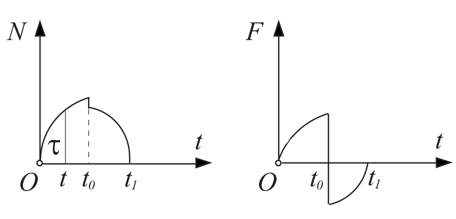

Fig. 2 Schematic time variation of the normal and friction components of the reactive force during the impact of duration t_1. The normal impulse up to an arbitrary time *t* is $\tau = \int_0^t N\mathrm{d}t$. The friction component of the reactive force is related to the normal component by the Amontons–Coulomb law of sliding friction $F = -\mu N\,\mathrm{sgn}(t - t_0)$, where t_0 is the time at which sliding changes its direction, and μ is the coefficient of kinetic friction.

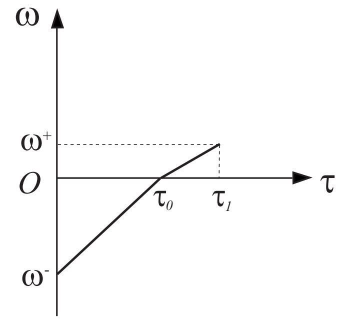

Fig. 3 Bilinear variation of the angular velocity $\omega = \omega(\tau)$, according to Eq. (9). The slopes in the compression and restitution phases of the impact are $(a+\mu b)/J_0$ and $(a-\mu b)/J_0$, respectively.

$$J_0(\omega - \omega^-) = a\tau + b\int_0^\tau \frac{F}{N}\mathrm{d}\tau \tag{5}$$

Let $0 < \tau_0 < \tau_1$ correspond to the instant t_0 when the angular velocity momentarily vanishes, $\omega(\tau_0) = 0$, and the slip reversal takes place at the transition between the compression and restitution phases of the impact. Assuming that during the impact the tangential component of the reactive force is related to the normal component by the Amontons–Coulomb law of sliding (dry) friction, and ignoring the tangential compliance of the colliding bodies, we can write

$$\frac{F}{N} = -\mu\,\mathrm{sgn}(\omega) = -\mu\,\mathrm{sgn}(\tau - \tau_0) \tag{6}$$

where μ is the coefficient of kinetic friction. The substitution into Eq. (5), upon integration, gives

$$\omega = \omega^- + \frac{a + \mu b}{J_0}\tau, \quad 0 \le \tau \le \tau_0$$

$$\omega = \frac{a - \mu b}{J_0}(\tau - \tau_0), \quad \tau_0 \le \tau \le \tau_1 \tag{7}$$

The normal impulse τ_0, determined from the condition $\omega(\tau_0) = 0$, is obtained from the first equation in Eq. (7), as

$$\tau_0 = -\frac{J_0\omega^-}{a + \mu b} \tag{8}$$

In order that $\omega > 0$ in the interval $(\tau_0, \tau_1]$, the coefficient of kinetic friction must be bounded by $\mu < a/b$. If $\mu \ge a/b$, the pendulum sticks to the ground after the impact, with no rebounding velocity.[1] By incorporating Eq. (8), the angular velocity expression (7) can be rewritten in a bilinear form (Fig. 3)

$$\omega = \left(1 - \frac{\tau}{\tau_0}\right)\omega^-, \quad 0 \le \tau \le \tau_0$$

$$\omega = \frac{a - \mu b}{a + \mu b}\left(1 - \frac{\tau}{\tau_0}\right)\omega^-, \quad \tau_0 \le \tau \le \tau_1 \tag{9}$$

The corresponding slopes $\mathrm{d}\omega/\mathrm{d}\tau$ in the compression and restitution phases of the impact are $(a+\mu b)/J_0$ and $(a-\mu b)/J_0$, respectively. The two slopes are equal only in the absence of friction.

3 Coefficient of Normal Restitution

The total normal impulse τ_1 is still an unknown quantity in the analysis, and cannot be determined without further assumptions.

[1]For small angle $\varphi = \arctan(a/b)$ there is no rebound if friction is sufficiently large ($\mu \ge \tan\varphi$); see Ref. [8], p. 180.

To proceed, we introduce the coefficient of normal restitution by the Poisson definition as the ratio of the normal impulses corresponding to restitution and compression phases of the impact, i.e.,

$$\kappa = \frac{\tau_1 - \tau_0}{\tau_0} > 0, \quad \tau_1 = (1+\kappa)\tau_0 \tag{10}$$

Since the duration of the impact and the variation of the normal force during the impact depend on friction, the ratio τ_1/τ_0 and thus the coefficient κ also depend on friction.[2] Assuming κ to be given, by using Eq. (10) it follows from Eq. (9) that the rebounding angular velocity $\omega^+ = \omega(\tau_1)$ is related to the incidence angular velocity by

$$\omega^+ = -\kappa \frac{a - \mu b}{a + \mu b} \omega^- \tag{11}$$

With this, the angular velocity during the restitution phase of the impact can be written, from the second equation in Eq. (9), as

$$\omega = \frac{1}{\kappa}\left(1 - \frac{\tau}{\tau_0}\right)\omega^+, \quad \tau_0 \leq \tau \leq \tau_1 \tag{12}$$

Introducing the horizontal and vertical velocity components of the contact point during the impact, $u = b\omega$ and $v = a\omega$, Eq. (11) can be rewritten as

$$v^+ + \mu u^+ = -\kappa(v^- - \mu u^-) \tag{13}$$

which demonstrates that in the case of rigid pendulum striking a fixed surface, the Poisson definition of the coefficient of normal restitution differs from the Newton definition[3]

$$\hat{\kappa} = -\frac{v^+}{v^-} = -\frac{\omega^+}{\omega^-} \tag{14}$$

Evidently, by using Eq. (11),

$$\hat{\kappa} = \frac{a - \mu b}{a + \mu b}\kappa = \frac{1 - \mu b/a}{1 + \mu b/a}\kappa \tag{15}$$

The two coefficients of normal restitution are therefore related to the coefficient that depends on the impact configuration, as represented by the ratio b/a, and the coefficient of friction μ. A physical interpretation of the parameter $\mu b/a$ will be further discussed in Sec. 6.1.

In order that there is a rebound, ω^- and ω^+ have to be of the opposite sign, so that $\hat{\kappa} > 0$. Thus, the coefficient of friction and the geometric parameters of the impact configuration (a, b) have to be such that $a - \mu b > 0$. If $\mu \geq a/b$, the pendulum sticks to the ground after the impact ($\omega^+ = 0$), which establishes the lower bound on Newton's coefficient of normal restitution, $\hat{\kappa} = 0$.

The kinetic energy dissipated by the frictional impact is

$$\Delta E = E^- - E^+ = \tfrac{1}{2}J_0[(\omega^-)^2 - (\omega^+)^2] = \tfrac{1}{2}(1 - \hat{\kappa}^2)J_0(\omega^-)^2 \tag{16}$$

i.e., in view of Eq. (15),

$$\Delta E = \frac{1}{2}J_0(\omega^-)^2\left[1 - \kappa^2\frac{(a - \mu b)^2}{(a + \mu b)^2}\right] \tag{17}$$

Since ΔE must be non-negative ($\Delta E \geq 0$, the equality holding if and only if $\kappa = 1$ and $\mu = 0$), and since $\kappa > 0$, Eq. (17) imposes an upper bound on Poisson's coefficient of normal restitution

$$\kappa \leq \kappa_{\max}, \quad \kappa_{\max} = \frac{a + \mu b}{a - \mu b} \tag{18}$$

for all cases in which there is a rebound of the pendulum after the impact ($a - \mu b > 0$). Since $\kappa_{\max} > 1$, the coefficient of normal restitution based on the Poisson definition can be greater than 1. This is also clear from relationship (15) between Poisson's and Newton's definitions of the coefficient of normal restitution, because $\hat{\kappa} \leq 1$ in order that $\Delta E = (1 - \hat{\kappa}^2)E^- \geq 0$, and thus $\kappa_{\max} > 1$. Lower values of the upper bounds on κ and $\hat{\kappa}$ will be derived in Sec. 5 of this paper.

4 Energetic Coefficient of Restitution

Stronge [5] introduced an energetic coefficient of normal restitution, whose square is equal to the negative ratio of the elastic strain energy released during restitution and the internal energy of deformation absorbed during compression phase of the impact. If tangential compliance of colliding bodies is negligible, this coefficient equals the negative ratio of the work done by the normal component of the impulsive reaction during restitution and compression phases of the impact,

$$\eta^2 = -\frac{W_r^n}{W_c^n} \tag{19}$$

where

$$W_c^n = \int_0^{t_0} Nv\,\mathrm{d}t = \int_0^{\tau_0} v\,\mathrm{d}\tau, \quad W_r^n = \int_{t_0}^{t_1} Nv\,\mathrm{d}t = \int_{\tau_0}^{\tau_1} v\,\mathrm{d}\tau \tag{20}$$

The ratio $-W_r^n/W_c^n$ accounts for irreversible deformation in the contact region, it is presumingly independent of friction, and thus represents an appealing coefficient to account for the normal restitution during a frictional impact.[4] For the pendulum striking a fixed surface, the vertical velocity component of the contact point is $v = a\omega$, so that Eq. (20) becomes

$$W_c^n = a\int_0^{\tau_0} \omega\,\mathrm{d}\tau, \quad W_r^n = a\int_{\tau_0}^{\tau_1} \omega\,\mathrm{d}\tau \tag{21}$$

By using angular velocity expression (7), this gives

$$W_c^n = \tfrac{1}{2}a\tau_0\omega^-, \quad W_r^n = \tfrac{1}{2}\kappa a\tau_0\omega^+ \tag{22}$$

When Eq. (22) is incorporated into Eq. (19), there follows

$$\eta^2 = -\kappa\frac{\omega^+}{\omega^-} = \kappa\hat{\kappa} = \frac{a - \mu b}{a + \mu b}\kappa^2 = \frac{a + \mu b}{a - \mu b}\hat{\kappa}^2 \tag{23}$$

The expression $\eta^2 = \kappa\hat{\kappa}$, derived by a different route, was first reported in Ref. [7]. Thus, for the rigid pendulum striking a fixed surface, the energetic coefficient of normal restitution is a geometric mean of the Newton and Poisson coefficients of normal restitution, i.e.,

$$\eta = \sqrt{\kappa\hat{\kappa}} \tag{24}$$

In the case of frictionless impact ($\mu = 0$), the three coefficients of normal restitution are equal to each other ($\kappa = \hat{\kappa} = \eta$). For frictional impact, the energetic coefficient is smaller than Poisson's and greater than Newton's coefficient of normal restitution ($\hat{\kappa} < \eta < \kappa$). Indeed, from Eq. (23),

$$\kappa = \left(\frac{a + \mu b}{a - \mu b}\right)^{1/2}\eta = \left(1 + \frac{2\mu b}{a - \mu b}\right)^{1/2}\eta$$

[2]The coefficient κ also depends on the material properties and the incidence angular velocity ω^-, affecting the nature of the deformation, but such dependence cannot be determined within a rigid body mechanics [10].

[3]The two definitions yield different expressions in many (but not all) frictional impact problems in which there is a change in the slip direction during the impact process. For example, Newton's and Poisson's definitions of the coefficient of normal restitution, as well as the energetic definition, to be discussed in Sec. 4, for a spinning disk striking a rough horizontal surface are all equivalent ($\kappa = \hat{\kappa} = \eta$).

[4]From experimental or numerical finite element method evaluations, it may be anticipated that η depends on the material properties, the radius of pendulum's local curvature in the contact region, and the incidence angular velocity affecting the extent of inelastic deformation in the region of contact.

$$\hat{\kappa}=\left(\frac{a-\mu b}{a+\mu b}\right)^{1/2}\eta=\left(1-\frac{2\mu b}{a-\mu b}\right)^{1/2}\eta \tag{25}$$

For example, if $\mu=0.1$ and $a/b=0.3$, one has $\kappa=\sqrt{2}\eta$ and $\hat{\kappa}=\eta/\sqrt{2}$.

Since $\tau_1/\tau_0=1+\kappa$, and in view of Eq. (23), the relationship between the impulse ratio τ_1/τ_0 and the coefficient η is

$$\frac{\tau_1}{\tau_0}=1+\sqrt{\frac{a+\mu b}{a-\mu b}}\eta, \quad a-\mu b>0 \tag{26}$$

In an advanced treatise on impact mechanics [8], there is a mistake in the derivation presented in pages 178–180, where the impulse ratio and the energy coefficient of restitution are listed as[5]

$$\frac{p_f}{p_c}=1+e_*\sqrt{\frac{r_1+\mu r_3}{r_1-\mu r_3}}, \quad e_*^2=\frac{r_1-\mu r_3}{r_1+\mu r_3}\frac{p_f^2-p_c^2}{p_c^2}$$

For example, if $\mu=0$, the first of these expressions yields $1+e_*=p_f/p_c$ and the second $1+e_*^2=p_f^2/p_c^2$, contradicting each other. The expression for e_*^2 in Ref. [8] is incorrect because of the mistake in the expression for the vertical velocity component used therein to evaluate the restitution work.

5 Bounds on the Coefficients of Restitution

An obvious upper bound on Newton's coefficient of normal restitution, appearing in the relationship $\omega^+=-\hat{\kappa}\omega^-$, is

$$\hat{\kappa}\leq 1 \tag{27}$$

because the magnitude of ω^+ cannot be greater than the magnitude of ω^- (otherwise there would be an energy gain by the impact process $E^+-E^->0$). In view of relationship (15), the corresponding upper bound on the Poisson coefficient of normal restitution is

$$\kappa\leq\frac{a+\mu b}{a-\mu b} \tag{28}$$

Lower upper bounds on κ and $\hat{\kappa}$ can be deduced by first imposing the upper bound on the energetic coefficient of normal restitution,

$$\eta\leq 1 \tag{29}$$

which must hold because the restitution phase of the impact cannot deliver more energy than what was stored during the compression phase, $\eta^2=-W_r^n/W_c^n\leq 1$. The limiting case $\eta=1$ corresponds to purely elastic compression (dissipation of energy being associated with the frictional sliding only). Consequently, by recalling from Eq. (25) the relationships between the three coefficients of normal restitution,[6] inequality (29) yields the stronger (lower) upper bounds on the Poisson and Newton coefficients of normal restitution. For $a-\mu b>0$, these are

$$\kappa\leq\left(\frac{a+\mu b}{a-\mu b}\right)^{1/2}=\left(1+\frac{2\mu b}{a-\mu b}\right)^{1/2}$$

$$\hat{\kappa}\leq\left(\frac{a-\mu b}{a+\mu b}\right)^{1/2}=\left(1-\frac{2\mu b}{a+\mu b}\right)^{1/2} \tag{30}$$

In the presence of friction, the above upper bound on κ is greater than 1, and the upper bound on $\hat{\kappa}$ is smaller than 1, the two being the reciprocals of each other.

The corresponding upper bound on the normal impulse $\tau_1=(1+\kappa)\tau_0$ is

$$\tau_1\leq\left[1+\left(1+\frac{2\mu b}{a-\mu b}\right)^{1/2}\right]\tau_0, \quad a-\mu b>0 \tag{31}$$

which shows that, for frictional impact, the upper bound on τ_1 is greater than $2\tau_0$. For frictionless impact, the upper bound on τ_1 is equal to $2\tau_0$, and is reached in the limit of perfectly elastic impact.

6 Energy Dissipated by Friction

The works done by the tangential component of impulse during the restitution and compression phases are

$$W_c^t=\int_0^{t_0}Fu\mathrm{d}t=\int_0^{\tau_0}\frac{F}{N}u\mathrm{d}\tau$$

$$W_r^t=\int_{t_0}^{t_1}Fu\mathrm{d}t=\int_{\tau_0}^{\tau_1}\frac{F}{N}u\mathrm{d}\tau \tag{32}$$

Since $F/N=-\mu\,\mathrm{sgn}(\tau-\tau_0)$, and by using the expression for the horizontal velocity component of the contact point $u=b\omega$, Eq. (32) becomes

$$W_c^t=\mu b\int_0^{\tau_0}\omega\mathrm{d}\tau, \quad W_r^t=-\mu b\int_{\tau_0}^{\tau_1}\omega\mathrm{d}\tau \tag{33}$$

i.e.,

$$W_c^t=\tfrac{1}{2}\mu b\tau_0\omega^-, \quad W_r^t=-\tfrac{1}{2}\mu b\kappa\tau_0\omega^+ \tag{34}$$

The total work dissipated by friction during the impact is

$$W^t=W_c^t+W_r^t=\tfrac{1}{2}\mu b\tau_0(\omega^--\kappa\omega^+) \tag{35}$$

This can be compared with the total work done by the normal component of impulse, which is, from Eq. (22),

$$W^n=W_c^n+W_r^n=\tfrac{1}{2}a\tau_0(\omega^-+\kappa\omega^+) \tag{36}$$

The dissipated energy by irreversible deformation due to normal force is

$$-W^n=-\tfrac{1}{2}a\tau_0\omega^-(1-\kappa\hat{\kappa})=\tfrac{1}{2}J_0(\omega^-)^2\frac{a}{a+\mu b}(1-\eta^2) \tag{37}$$

which is positive if $\eta<1$. See also a related discussion in Ref. [11].

6.1 Work Ratios. The total works done during the compression and restitution phases of the impact are

$$W_c=W_c^n+W_c^t=\tfrac{1}{2}(a+\mu b)\tau_0\omega^-=-\tfrac{1}{2}J_0(\omega^-)^2 \tag{38}$$

$$W_r=W_r^n+W_r^t=\tfrac{1}{2}(a-\mu b)\kappa\tau_0\omega^+=\tfrac{1}{2}J_0(\omega^+)^2 \tag{39}$$

These expressions can also be obtained directly by applying the energy-work principle to the compression and restitution phases of the impact separately ($E^-+W_c=0$ and $0+W_r=E^+$). The total work done by both the normal and tangential impulsive reactions is $W=W_c+W_r=W^n+W^t=E^+-E^-$.

From the derived work expressions, the following work ratios are observed:

$$-\frac{W_r^n}{W_c^n}=\frac{W_r^t}{W_c^t}=-\kappa\frac{\omega^+}{\omega^-}=\kappa\hat{\kappa}=\eta^2, \quad -\frac{W_r}{W_c}=\left(\frac{\omega^+}{\omega^-}\right)^2=\hat{\kappa}^2 \tag{40}$$

Thus, for the pendulum striking a fixed surface, the Newton coefficient of normal restitution, defined by the kinematic relation $\hat{\kappa}=-\omega^+/\omega^-$, can also be given an energy interpretation, via $\hat{\kappa}^2=-W_r/W_c=E^+/E^-$. Furthermore, the energetic coefficient of normal restitution is equal to either the normal work ratio $(-W_r^n/W_c^n)$, or the frictional work ratio (W_r^t/W_c^t), so that in the considered problem the frictional dissipation during the restitution phase is always

[5] The notations used in Ref. [8] are $p_c=\tau_0$, $p_f=\tau_1$, $e_*=\eta$, $r_1=a$, and $r_3=b$.

[6] If η is assumed to be given, then the kinematic and kinetic coefficients of normal restitution depend on η and the geometric parameter $\mu b/a$, accounting for friction and the pendulum impact configuration, represented by the ratio b/a. Clearly, the higher the value of η (less dissipation by the deformation), the higher the coefficients κ and $\hat{\kappa}$, and thus the higher the rebound (ω^+).

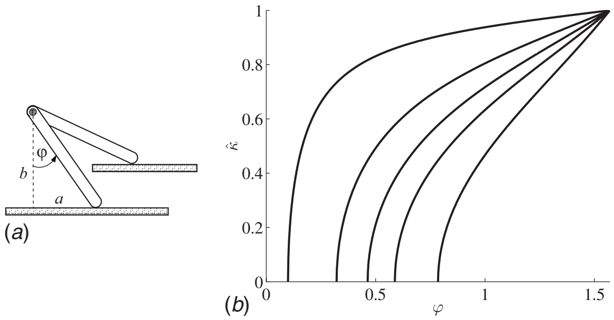

Fig. 4 (*a*) Two impact configurations of the rigid pendulum against a fixed surface, corresponding to two different values of the angle φ. (*b*) The variation of the Newton coefficient of normal restitution $\hat{\kappa}$ with φ (in radians), in the case of an elastic impact ($\eta=1$), with different coefficients of friction μ. The far left curve is for $\mu=0.1$, and the subsequent curves toward the right are for $\mu=1/3,0.5,2/3,1$. As $\varphi\to\pi/2$, the coefficient $\hat{\kappa}=-\omega^+/\omega^-\to 1$ for all μ (passive friction for vertical impact).

smaller than during the compression phase ($W_r^t<W_c^t$ for $\eta<1$).

The ratio of the works associated with the tangential and normal impulses is

$$\frac{W^t}{W^n}=\frac{\mu b}{a}\frac{1+\eta^2}{1-\eta^2} \tag{41}$$

while

$$\frac{W_c^t}{W_c^n}=-\frac{W_r^t}{W_r^n}=\frac{\mu b}{a} \tag{42}$$

Thus, the parameter $\mu b/a$, appearing in the relationships between κ, $\hat{\kappa}$, and η, can be given a physical interpretation as energy ratio (42), in addition to being equal to the product of the force ratio $|F|/N$ and the velocity ratio u/v.

If, upon the impact, the pendulum sticks to the ground, there is no restitution ($W_r^t=W_r^n=0$), and $W_c^n=W_c^t=\tau_0 a\omega^-/2=-E^-/2$. In this case, all three coefficients of normal restitution are equal to zero, which represents their lower bound, and $\tau_1=\tau_0=-J_0\omega^-/(2a)$.

7 Impact With Elastic Compression

If $\eta=1$ in Eq. (40), then $W^n=0$ and $W_r^n=-W_c^n$, which corresponds to elastic deformation during the impact: The elastic energy stored during the compression phase is fully recovered and used for the liftoff of the pendulum during the restitution phase.[7] In this case, from Eq. (25), the Newton and Poisson coefficients of restitution are the reciprocals of each other ($\hat{\kappa}=1/\kappa$), and equal to

$$\kappa=\left(\frac{a+\mu b}{a-\mu b}\right)^{1/2}=\left(1+\frac{2\mu b}{a-\mu b}\right)^{1/2}>1$$

$$\hat{\kappa}=\left(\frac{a-\mu b}{a+\mu b}\right)^{1/2}=\left(1-\frac{2\mu b}{a+\mu b}\right)^{1/2}<1 \tag{43}$$

Thus, for the pendulum striking a rough surface elastically, Newton's coefficient of normal restitution is smaller than 1, and Poisson's coefficient is greater than 1. The plots of $\hat{\kappa}$ versus the angle $\varphi=\arctan(a/b)$, for various values of μ, are shown in Fig. 4. The plots are for the elastic impact. For each μ, there is no rebound if the angle φ is smaller than the angle corresponding to $\hat{\kappa}=0$. If the frictional impact is inelastic, the plots should be scaled by the corresponding value of $\eta<1$.

[7]The tangential stiffnesses of the pendulum and the surface are assumed to be infinite, so that the elastic energy is entirely due to deformation in the normal direction.

The total normal impulse of the elastic impact is

$$\tau_1=\left[1+\left(1+\frac{2\mu b}{a-\mu b}\right)^{1/2}\right]\tau_0,\quad a-\mu b>0 \tag{44}$$

which is greater than $2\tau_0$ (unless $\mu=0$, in which case $\tau_1=2\tau_0$).

The average angular velocity during the elastic impact is zero, and

$$\omega^+=-\hat{\kappa}\omega^-=-\frac{\omega^-}{\kappa}=-\left(\frac{a-\mu b}{a+\mu b}\right)^{1/2}\omega^- \tag{45}$$

The total work done by the impulsive reactions is equal to the dissipated work by friction, which is, from Eq. (35),

$$W=W^t=\tfrac{1}{2}\mu b\tau_0(1+\kappa^2)(\omega^-)^2=-J_0\frac{\mu b}{a+\mu b}(\omega^-)^2 \tag{46}$$

8 Tangential Impact Coefficient

The tangential impulse at an arbitrary stage of the impact process is

$$f(\tau)=\int_0^t F\mathrm{d}t=\int_0^\tau \frac{F}{N}\mathrm{d}\tau \tag{47}$$

Since $F=-\mu N\,\mathrm{sgn}(\tau-\tau_0)$, one readily finds that

$$f(\tau)=\begin{cases}\mu\tau, & \tau\le\tau_0\\ \mu(2\tau_0-\tau), & \tau\ge\tau_0\end{cases} \tag{48}$$

Corresponding Routh's impact diagram is shown in Fig. 5.

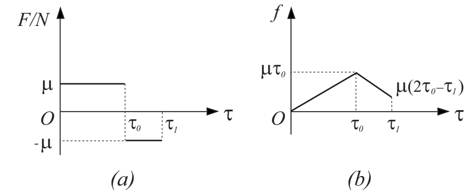

Fig. 5 (*a*) The force ratio *F*/*N* versus the normal impulse τ. (*b*) Routh's impact diagram showing the variation of the tangential impulse *f* versus the normal impulse τ.

Brach [6] defined the tangential impact coefficient as the ratio of the tangential and normal components of the impulse,

$$\hat{\mu}=\frac{\int_0^{t_1}F\mathrm{d}t}{\int_0^{t_1}N\mathrm{d}t}=\frac{f_1}{\tau_1} \tag{49}$$

For the rigid pendulum striking a fixed surface, this gives

$$\hat{\mu}=\frac{\mu(2\tau_0-\tau_1)}{\tau_1} \tag{50}$$

Since $\tau_1=(1+\kappa)\tau_0$, and thus $f_1=\mu(1-\kappa)\tau_0$, the substitution into Eq. (50) yields

$$\hat{\mu}=\frac{\mu(1-\kappa)}{1+\kappa} \tag{51}$$

as in Ref. [4]. The ratio of the tangential and normal components of the impulse is thus not equal to μ (as in Whittaker's theory of frictional impact [2]), but to $\hat{\mu}<\mu$, because there was a change in the slip direction at the transition between the compression and restitution phases of the impact [4,7,8]. If there is no rebound ($\omega^+=0$), then $\kappa=0$ and $\hat{\mu}=\mu$ (as expected, because, without re-

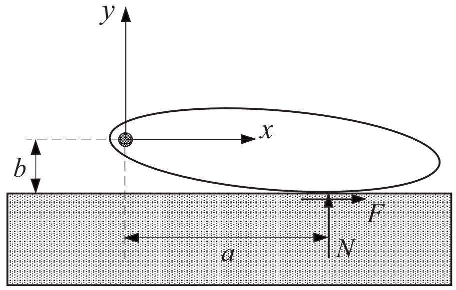

Fig. 6 A nearly vertical impact ($b \ll a$) of a rigid pendulum against a rough horizontal surface. The componential reactions at the contact point are *N* and *F*.

bound, there is no change in the slip direction). Note also that in Eq. (51), $\hat{\mu}<0$ if $\kappa>1$. If $\mu=0$, then $\hat{\mu}=0$, as well.

The energy dissipated by the impact $\Delta E=J_0[(\omega^-)^2-(\omega^+)^2]/2$ can be cast in the form of the generalized Thomson–Tait formula (e.g., Refs. (7) and (8))

$$\Delta E = -\tfrac{1}{2}f_1(u^- + u^+) - \tfrac{1}{2}\tau_1(v^- + v^+) \tag{52}$$

Indeed, from Eq. (1), one can write $a\tau_1+bf_1=-J_0(\omega^- - \omega^+)$, so that

$$\Delta E = \tfrac{1}{2}J_0(\omega^- - \omega^+)(\omega^- + \omega^+) = -\tfrac{1}{2}(a\tau_1 + bf_1)(\omega^- + \omega^+) \tag{53}$$

Since $u=b\omega$ and $v=a\omega$, Eq. (53) takes Thomson–Tait form (52).

8.1 Bounds on the Tangential Impact Coefficient. Since Poisson's coefficient of normal restitution is bounded by

$$0 \le \kappa \le \left(\frac{a+\mu b}{a-\mu b}\right)^{1/2} \tag{54}$$

it readily follows from Eq. (51) that the bounds on $\hat{\mu}$ are

$$-\frac{a}{b}\left[1-\sqrt{1-(\mu b/a)^2}\right] \le \hat{\mu} \le \mu \tag{55}$$

The upper bound $\hat{\mu}=\mu$ is reached if the pendulum sticks to the ground upon the impact ($\kappa=0$), while the lower bound is reached in the case of elastic frictional impact ($\eta=1$), since then κ is equal to its upper bound. Note that the lower bound on $\hat{\mu}$ in Eq. (55) is negative, but greater than $-a/b$ (which itself must be greater than $-\mu$, for the rebound to take place). For small ratio $\mu b/a$, the lower bound on $\hat{\mu}$ is approximately equal to $-\mu/2$. The negative values of $\hat{\mu}$ mean that in these cases there is a longer lasting backward than forward slip during the impact, i.e., the duration of the restitution phase is longer (on the τ-scale) than that of the compression phase ($\tau_1-\tau_0>\tau_0$, and thus $\kappa>1$). For example, in the case of an elastic impact ($\eta=1$) with $\mu=0.1$ and $a/b=0.3$, so that $\kappa=\sqrt{2}$, one finds that $\tau_1-\tau_0=\sqrt{2}\tau_0$ and $\hat{\mu}\approx-0.017$.

9 Nearly Vertical Impact

The effect of friction on the impact response diminishes with the decrease in the ratio b/a. For a given coefficient of friction μ, and sufficiently small ratio b/a (nearly vertical impact, Fig. 6), from Eq. (23) there follows[8]

$$\kappa = (1+\mu b/a)\eta, \quad \hat{\kappa} = (1-\mu b/a)\eta$$

$$\kappa = (1+2\mu b/a)\hat{\kappa}, \quad \hat{\kappa} = (1-2\mu b/a)\kappa \tag{56}$$

to first order terms in $\mu b/a$ (neglecting the quadratic term $(\mu b/a)^2 \ll 1$). In this case, and to this order of accuracy, the bounds on the kinematic and kinetic coefficients of normal restitution are

$$0 \le \hat{\kappa} \le 1-\mu b/a, \quad 0 \le \kappa \le 1+\mu b/a \tag{57}$$

The corresponding bounds on the total normal impulse are

$$\tau_0 \le \tau_1 \le (2+\mu b/a)\tau_0 \tag{58}$$

The tangential impact coefficient can be expressed in terms of η as

$$\hat{\mu} = \mu\left[\frac{1-\eta}{1+\eta} - 2\mu\frac{b}{a}\frac{\eta}{(1+\eta)^2}\right] \tag{59}$$

to first order terms in $\mu b/a$, and is bounded by

$$-\frac{1}{2}\mu^2\frac{b}{a} \le \hat{\mu} \le \mu \tag{60}$$

For example, if $\mu=0.4$ and $a=4b$, so that $\mu b/a=0.1$, the bounds on the impact coefficients are $0\le\kappa\le1.1$, $0\le\hat{\kappa}\le0.9$, and $-0.02 \le\hat{\mu}\le0.4$.

10 Conclusion

The frictional impact of rigid pendulum against a fixed surface was studied by using kinematic, kinetic, and energetic definitions of the coefficient of normal restitution, which specify the rebounding angular velocity and the total normal impulse during the impact. The tangential stiffnesses of the pendulum and the surface are assumed to be infinite, so that the elastic energy is entirely due to deformation in the normal direction. It is shown that the energetic coefficient of normal restitution is a geometric mean of the kinematic (Newton's) and kinetic (Poisson's) coefficients of normal restitution. In the case of frictionless impact, the three coefficients are equal to each other. For frictional impact, the energetic coefficient is smaller than Poisson's and greater than Newton's coefficient of normal restitution. The upper bounds on all three coefficients are established, demonstrating that, for the frictional impact, the upper bound on the Newton coefficient is smaller than 1, while the upper bound on the Poisson coefficient is greater than 1. The upper bound on the energetic coefficient of normal restitution is equal to 1, because the restitution phase of the impact cannot deliver more energy than what is stored during the compression phase. The frictional dissipation during the restitution phase is always smaller than during the compression phase. For the pendulum striking a rough surface elastically, without dissipation due to deformation (dissipation of energy being associated with the frictional sliding only), the Newton and Poisson coefficients are the reciprocals of each other. If, upon the impact, the pendulum sticks to the ground, there is no restitution phase of the impact, and all three coefficients of normal restitution are equal to zero, which represents their lower bound. The bounds on the tangential impact coefficient, defined by the ratio of the frictional and normal impulses (which is not equal to μ, because of the slip reversal at the transition from the compression to restitution phases of the impact), are also derived. Its lower bound is negative, while its upper bound is equal to the kinetic coefficient of friction. The simplified bounds on the impact coefficients are deduced in the case of a nearly vertical impact, for which friction exerts the least effect on the impact response. The obtained results may be of interest for the analysis of planar impacts of linkages and related problems in the mechanics of frictional impact [12–14].

Acknowledgment

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[8]Thus, η is here the arithmetic mean of κ and $\hat{\kappa}$, i.e., $\eta=(\kappa+\hat{\kappa})/2$, which is in agreement with the exact geometric mean relationship $\eta=(\kappa\hat{\kappa})^{1/2}$, to first order terms in $\mu b/a$.

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