MAE105
Final Exam
(open book, closed notes)

Name:_________________________

Time: 3:05 to 6:05pm

Date: December 08, 2005

Problem 1

(a) (2 Points) Find a general expression x = x(t), for the characteristics of the following PDE:

$$\frac{\partial u}{\partial t} + \tfrac{1}{2}\, x\, t\, \frac{\partial u}{\partial x} = \frac{2}{u}.$$

[Note that your expression must include a constant of integration, say, $x(0) = x_0$.]

(b) (0.5 Point) In the x, t-plane, $t > 0$, sketch the characteristic curve that starts from $x_0 = 1$ at $t = 0$.

(c) (2 Points) Find the general solution of this PDE.

(d) (1.5 Points) Specialize the solution in (b) such that at $t = 0$, we have $u(x_0, 0) = u_0 = (x_0 + 1)^{\frac{1}{2}}$.

Problem 2

Consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 \qquad (1)$$

in a finite domain, $0 < x < 1$, $t > 0$, with the boundary conditions

$$u(0,\ t) = u(1,\ t) = 0,$$

and initial conditions

$$u(x, 0) = f(x) = \begin{cases} 3x^2 & \text{for} \quad 0 < x < 1/2 \\ 1 - x^2 & \text{for} \quad 1/2 < x < 1. \end{cases}$$

$$\frac{\partial u}{\partial t}(x, 0) = g(x) = x(1 - x), \qquad 0 < x < 1.$$

(a) (1 Point) Draw the (x, t)-plane, and below the x-axis, draw the initial conditions in two graphs, as discussed in the class and in your book.

(b) (2 Points) Extend the initial conditions such that the boundary conditions are satisfied for all $t > 0$, if we use the general solution for the infinite domain. Sketch the extended initial conditions for $-3 < x < 3$.

(c) (2 Points) In the x, t-plane, draw the characteristics that pass through the following points:

$[x = 0, t = 1/4]$,
$[x = 1/2, t = 1/2]$,
$[x = 1/4, t = 4/3]$.

(d) (1 Point) Write down the general solution of the wave equation in an infinite domain for PDE (1).

(e) (3 Points) Use your general solution and the characteristics to find the value of u(x, t) for the following indicated values of x and t:

$[x = 0, t = 1/4]$,
$[x = 1/2, t = 1/2]$,
$[x = 1/4, t = 4/3]$.

Problem 3

(a) (1 Point) Expand $f(x) = \cos x$, $0 < x < \pi$, in Fourier sine series and find all the coefficients explicitly.

(b) (1 Point) Expand $g(x) = \sin x$, $0 < x < \pi$, in Fourier cosine series and find all the coefficients explicitly.

(c) (0.5 Point) Explain which representation above is an odd extension and which is an even extension of the corresponding function. Sketch the extended functions, f(x) and g(x), for $-2\pi < x < 2\pi$.

(d) (0.5 Point) Explain which of the two Fourier series can be differentiated term by term to obtain the Fourier series representation of the derivative of the original function, and why.

Problem 4

Consider the following nonhomogeneous diffusion equation:

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + e^t (1 - x) = 0, \qquad 0 < x < \pi, \quad t > 0, \tag{2}$$

with boundary conditions

$$u(0, t) = 0, \qquad u(\pi, t) = 0, \tag{3}$$

and the initial condition

$$u(x, 0) = \cos x. \tag{4}$$

(a) (1.5 Points) Consider the following eigenvalue problem:

$$\frac{d^2\phi}{dx^2} + \lambda^2 \phi = 0,$$

and write down the boundary conditions which are appropriate for solving the nonhomogeneous PDE (2). Find the eigenfunctions, say, $\phi_n(x)$, that can be used to solve (2) with boundary conditions (3).

(b) (1.5 Points) Expand $1 - x$ in the Fourier series of $\phi_n(x)$, i.e., set $1 - x \approx \sum_{n=1}^{\infty} B_n \phi_n(x)$, and calculate the coefficients B_n explicitly, where $\phi_n(x)$ are the eigenfunctions that you obtained in (a).

(c) (3 Points) Set $u(x, t) \approx \sum_{n=1}^{\infty} A_n(t) \phi_n(x)$ where $\phi_n(x)$ are the eigenfunctions that you obtained in (a), substitute this and the Fourier expansion of $1 - x$ into the original PDE (2), use orthogonality of the eigenfunctions, and reduce the PDE (2) into ODE's for the coefficients $A_n(t)$.

(d) (3 Points) Solve the ODE's to find $A_n(t)$; note that this solution must include an intergation constant, say, C_n. Find the integration constant using the Fourier expansion of the initial condition (4). Then obtain the complete series solution of (2) subject to (3) and (4).

Problem 5

(a) (0.5 Point) Write down the two dimensional Laplace's PDE in polar coordinates, i.e., in the r, θ-plane.

(b) (1 Point) Use separation of variables, $u(r, \theta) = \phi(\theta)\, f(r)$, to find an ODE for $\phi(\theta)$ and an ODE for $f(r)$, such that $\phi(\theta)$ is periodic in θ.

(c) (1 Point) Write down the general solution of the ODE of $\phi(\theta)$ that must include two integration constants.

(d) (1 Point) Noting that the ODE for $f(r)$ is *equidimentional*, find its general solution that must include two integration constants.

Consider now a section of a circular annulus defined by

$$0 < \theta < \pi / 3, \qquad a < r < b, \tag{5}$$

where a and b are constants. Consider also the following boundary conditions:

$$u(r, 0) = 0, \qquad u(r, \pi / 3) = 0, \tag{6}$$

$$u(a, \theta) = 0, \qquad u(b, \theta) = \sin 4\theta. \tag{7}$$

(e) (1.5 Point) Apply the boundary conditions (6) to find the corresponding boundary conditions for $\phi(\theta)$, and use these to find the associated eigenvalues and eigenfunctions.

(f) (1.5 Points) Write down the general infinite series solution of $u(r, \theta)$. Then apply the boundary conditions (7) to obtain two equations for the coefficients of this series.

(g) (1.5 Points) Find all the non-zero coefficients (using orthogonality) and write down the final solution for $u(r, \theta)$.