Wave Equation and Method of Characteristics

MAE 105

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Wave Equation in Infinite One Spatial-dimensional

Method of characteristics

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2}, \quad t > 0$$

Initial conditions:

$$u(x,0) = f(x)$$

$$\frac{\partial u}{\partial t}(x,0) = g(x)$$

Rewrite the wave equation as follows:

$$\left(\frac{\partial}{\partial t}+c\frac{\partial}{\partial x}\right)\left(\frac{\partial u}{\partial t}-c\frac{\partial u}{\partial x}\right)=0$$

$$\left(\frac{\partial}{\partial t}-c\frac{\partial}{\partial x}\right)\left(\frac{\partial u}{\partial t}+c\frac{\partial u}{\partial x}\right)=0$$

Set

$$w=\frac{\partial u}{\partial t}-c\frac{\partial u}{\partial x}$$

$$v=\frac{\partial u}{\partial t}+c\frac{\partial u}{\partial x}$$

Obtain

$$\frac{\partial w}{\partial t}+c\frac{\partial w}{\partial x}=0$$

$$\frac{\partial v}{\partial t}-c\frac{\partial v}{\partial x}=0$$

Set

$$\frac{dw\left(x(t),t\right)}{dt} = \frac{\partial w}{\partial t} + \frac{dx}{dt}\frac{\partial w}{\partial x}$$

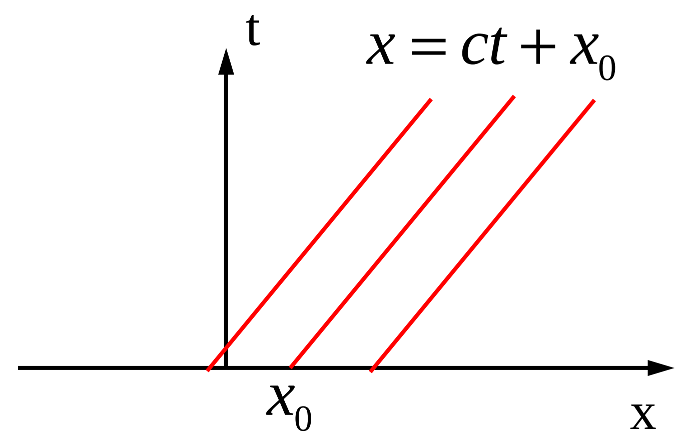
$$\frac{dx}{dt} = c \quad \Rightarrow \quad \frac{dw}{dt} = 0 \quad \text{on} \quad x = ct + x_0$$

With initial condition

$$w(x,0) = P(x)$$

we obtain

$$w(x,t) = P(x_0) = P(x - ct)$$



Also, set

$$\frac{dv\left(x(t),t\right)}{dt} = \frac{\partial v}{\partial t} - \frac{dx}{dt}\frac{\partial v}{\partial x}$$

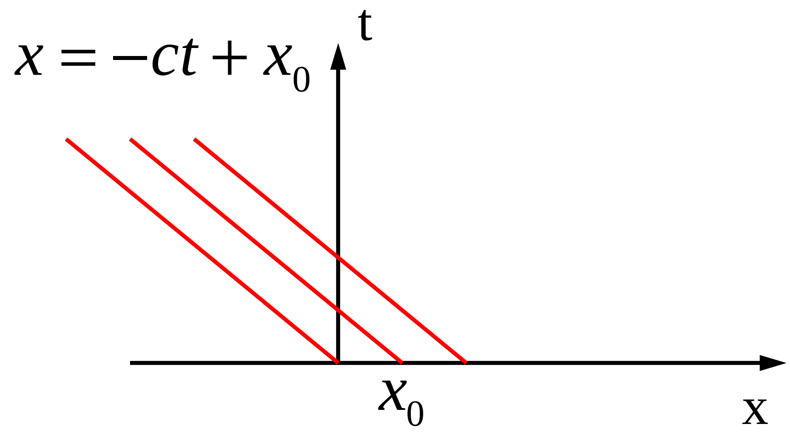
$$\frac{dx}{dt} = -c \quad \Rightarrow \quad \frac{dv}{dt} = 0 \quad \text{on} \quad x = -ct + x_0$$

With initial condition

$$v(x,0) = Q(x)$$

we obtain

$$v(x,t) = Q(x_0) = Q(x+ct)$$



General solution

$$u(x,t) = F(x-ct) + G(x+ct)$$

$$u(x,0) = f(x) \qquad -\infty < x < \infty$$

$$\frac{\partial u}{\partial t}(x,0) = g(x) \qquad -\infty < x < \infty$$

$$G(x) = \frac{1}{2} f(x) + \frac{1}{2c}\int_0^x g(\bar{x})\,d\bar{x} + k$$

$$F(x) = \frac{1}{2} f(x) - \frac{1}{2c}\int_0^x g(\bar{x})\,d\bar{x} - k$$

D'alembert's solution

$$u(x,t) = \frac{f(x-ct) + f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\bar{x})\,d\bar{x}$$

Semi-Infinite strings and reflections

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2}, \quad t > 0,\ x > 0$$

$$u(x,0) = f(x)$$

$$\frac{\partial u}{\partial t}(x,0) = g(x)$$

$$u(0,t) = 0$$

$$x > ct$$

$$u(x,t) = \frac{f(x-ct) + f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\bar{x})\,d\bar{x}$$

$$x < ct = z$$

$$u(0,t) = 0 \qquad\qquad F(z) = -G(-z)$$

$$u(x,t) = \frac{f(x+ct) - f(ct-x)}{2} + \frac{1}{2c}\int_{ct-x}^{x+ct} g(\bar{x})\,d\bar{x}$$

Finite strings and reflections

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2}, \quad t > 0, \; 0 < x < L$$

$$u(x,0) = f(x)$$

$$\frac{\partial u}{\partial t}(x,0) = g(x)$$

$$u(0,t) = 0$$

$$u(L,t) = 0$$

For

$$0 < x - ct < L \qquad 0 < x + ct < L$$

$$u(x,t) = \frac{f(x-ct) + f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\bar{x})\,d\bar{x}$$

For

$$0 > x - ct \qquad x + ct > L$$

$$u(0,t) = 0 \qquad F(-ct) = -G(ct)$$

$$u(L,t) = 0 \qquad F(L-ct) = -G(L+ct)$$

⇨ **Extend the initial conditions as *odd* functions**