MAE105 - Quiz #6 Solutions

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \sin x \qquad t>0, \quad 0<x<1$$

B.C.'s: $u(0,t)=1, \quad u(1,t)=2$

a) $\dfrac{d^2 w(x)}{dx^2} = -\sin x \quad , \quad w(0)=1 \ , \ w(1)=2$

$$\frac{dw(x)}{dx} = \cos x + A$$

$$w(x) = \sin x + Ax + B$$

$$w(0) = 1 = \cancel{\sin 0} + \cancel{A(0)} + B$$

$$B = 1$$

$$w(1) = 2 = \sin 1 + A(1) + 1$$

$$A = 1 - \sin(1)$$

$$\boxed{w(x) = \sin x + (1-\sin(1))x + 1}$$

b) $u(x,t) = v(x,t) + u_E(x)$

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + \underbrace{\frac{d^2 u_E}{dx^2} + \sin x}_{=0 \text{ (like part a)}} \Rightarrow \frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}$$

w(x) from part a

B.C.'s: $u(0,t) = v(0,t) + u_E(0) = 1$

$$v(0,t) + 1 = 1$$

$$v(0,t) = 0$$

$$u(1,t) = v(1,t) + u_E(1) = 2$$

$$v(1,t) + 2 = 2$$

$$v(1,t) = 0$$

$\therefore$ $\boxed{\begin{array}{l} \dfrac{d^2 u_E}{dx^2} = -\sin x \quad 0<x<1 \\ u_E(0)=1 \\ u_E(1)=2 \end{array}}$, $\boxed{\begin{array}{l} \dfrac{\partial v}{\partial t} = \dfrac{\partial^2 v}{\partial x^2} \quad t>0, \ 0<x<1 \\ v(0,t)=0 \\ v(1,t)=0 \end{array}}$

c) $v(x,t) = \phi(x)G(t)$

$$\phi \frac{dG}{dt} = G\frac{d^2\phi}{dx^2}$$

$$\frac{1}{G}\frac{dG}{dt} = \frac{1}{\phi}\frac{d^2\phi}{dx^2} = -\lambda^2$$

$\frac{dG}{dt} + \lambda^2 G = 0$ and $\frac{d^2\phi}{dx^2} + \lambda^2\phi = 0$

$G(t) = Ae^{-\lambda^2 t}$

$$\phi(x) = B\cos\lambda x + C\sin\lambda x$$

$$\phi(0) = 0 = B$$

$$\phi(1) = 0 = C\sin\lambda$$

$\boxed{\lambda_n = n\pi}$ for $n = 0, 1, 2, \ldots$

$$\boxed{\phi_n(x) = \sin n\pi x}$$

$$\boxed{v(x,t) = \sum_{n=1}^{\infty} A_n \sin(n\pi x)\, e^{-(n\pi)^2 t}}$$

d) I.C. : $u(x,0) = x = v(x,0) + u_E(x)$

$$v(x,0) = x - \left[\sin x + (1-\sin(1))x + 1\right]$$

$$\boxed{v(x,0) = \sin(1)x - \sin x - 1}$$

e) $v(x,0) = \sin(1)x - \sin x - 1 = \sum_{n=1}^{\infty} A_n \sin(n\pi x) \cancel{e^{0}}$

$$\int_0^1 (\sin(1)x - \sin x - 1)\sin(n\pi x)\,dx = A_n \int_0^1 \underbrace{\sin^2(n\pi x)}_{\frac{1}{2} - \cancel{\frac{\cos(2n\pi x)}{2}}}\,dx$$

$$\boxed{A_n = 2\int_0^1 (\sin(1)x - \sin x - 1)\sin(n\pi x)\,dx}$$

$$= A_n^{(1)} + A_n^{(2)} + A_n^{(3)}$$

Extra Point

$$A_n^{(1)} = 2\sin(1)\int_0^1 x\sin(n\pi x)\,dx$$

$$= 2\sin(1)\left(\left[-\frac{x}{n\pi}\cos(n\pi x)\right]_0^1 + \cancel{\int_0^1 \cos(n\pi x)\,dx}\right)$$

$$= -\frac{2\sin(1)}{n\pi}(-1)^n$$

$$A_n^{(2)} = 2\int_0^1 -\sin x \sin(n\pi x)\,dx$$
$$= -\int_0^1 \left(\cos(x+n\pi x) - \cos(x-n\pi x)\right)dx$$
$$= -\int_0^1 \cos(x(1+n\pi))\,dx - \int_0^1 \cos(x(1-n\pi))\,dx$$
$$= -\left[\frac{1}{1+n\pi}\sin(x(1+n\pi))\right]_0^1 - \left[\frac{1}{1-n\pi}\sin(x(1-n\pi))\right]_0^1$$
$$= -\frac{\sin(1+n\pi)}{1+n\pi} + \frac{\sin(1-n\pi)}{1-n\pi}$$

$$A_n^{(3)} = 2\int_0^1 -\sin(n\pi x)\,dx = 2\left[\frac{1}{n\pi}\cos(n\pi x)\right]_0^1 = \frac{2(-1)^n}{n\pi} - \frac{2}{n\pi}$$

$$u(x,t) = v(x,t) + u_E(x)$$

$$\boxed{\begin{aligned} u(x,t) = \sum_{n=1}^{\infty}\left(-\frac{2\sin(1)}{n\pi}(-1)^n - \frac{\sin(1+n\pi)}{1+n\pi} + \frac{\sin(1-n\pi)}{1-n\pi} + \frac{2(-1)^n}{n\pi} - \frac{2}{n\pi}\right)\sin(n\pi x)\,e^{-(n\pi)^2 t} \\ + \sin x + (1-\sin(1))\,x + 1 \end{aligned}}$$