SP09 MAE 105 Quiz #3 Solution 4/30/09

$$\frac{d^2\phi}{dx^2} + \lambda^2\phi = 0, \qquad 0 < x < \pi$$

(a) $\boxed{\phi(x) = A\cos(\lambda x) + B\sin(\lambda x)}$: general solution

(b) B.C.'s

i) $\phi(0) = \phi'(0) \Rightarrow A = -\lambda A \underbrace{\sin(\lambda\cdot 0)}_{\to 0} + \lambda B \underbrace{\cos(\lambda \cdot 0)}_{\to 1}$

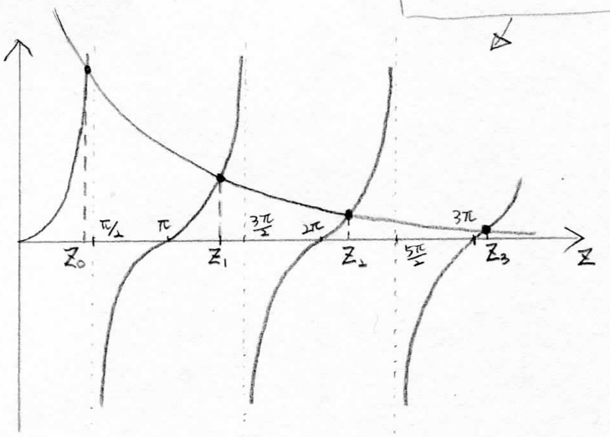
$\boxed{A = \lambda B}$

ii) $\phi'(\pi) = 0 \Rightarrow -\cancel{\lambda} A\sin(\lambda\pi) + \cancel{\lambda} B\cos(\lambda\pi) = 0.$

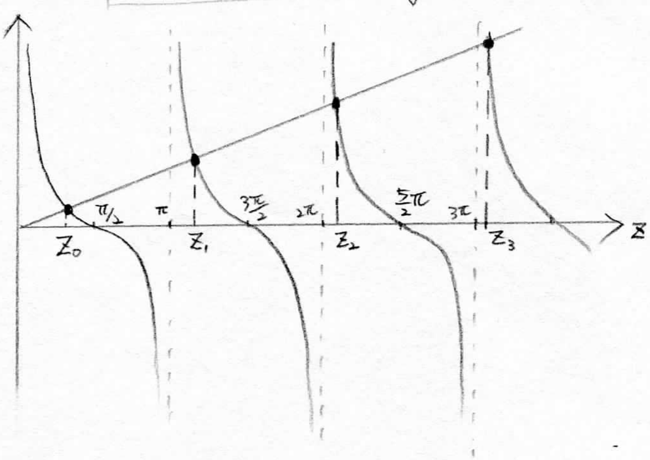
$\boxed{A\tan(\lambda\pi) = B}$

(c) $\lambda \cancel{B}\cdot\tan(\lambda\pi) = \cancel{B} \Rightarrow \boxed{\tan(\lambda\pi) = \frac{1}{\lambda}}$

(d) Letting $\lambda\pi = z$, $\boxed{\tan z = \frac{\pi}{z}}$ or $\boxed{\cot z = \frac{z}{\pi}}$



or



(e) $\lim_{n\to\infty} z_n = n\pi$ $\quad\therefore \boxed{\lim_{n\to\infty}\lambda_n = n}$ $\quad n = 0, 1, 2 \cdots$

(Extra 1) multiplying (1) by $\phi(x)$ and integrating it,

$$\int_0^\pi \phi \frac{d^2\phi}{dx^2}\, dx + \int_0^\pi \lambda^2 \phi^2\, dx = 0$$

$$\left[\phi \frac{d\phi}{dx}\right]_0^\pi - \int_0^\pi \left(\frac{d\phi}{dx}\right)^2 dx + \lambda^2 \int_0^\pi \phi^2\, dx = 0$$

$$\cancel{\phi(\pi)\phi'(\pi)}^{\,0} - \phi(0)\phi'(0) - \int_0^\pi \left(\frac{d\phi}{dx}\right)^2 dx + \lambda^2 \int_0^\pi \phi^2\, dx = 0$$

$$\therefore \lambda^2 = \left[\frac{\phi^2(0) + \int_0^\pi \left(\frac{d\phi}{dx}\right)^2 dx}{\int_0^\pi \phi^2\, dx}\right] \quad : \text{ the Rayleigh quotient}$$

Note that λ is not the Rayleigh quotient or the eigenvalue in this problem.

(Extra 2) Substituting $\phi_{appr.}(x) = \cos x - 1$,

$\phi(0) = 0$, $\left(\frac{d\phi}{dx}\right)^2 = (-\sin x)^2 = \sin^2 x$, and $\phi^2 = \cos^2 x - 2\cos x + 1$

$$\int_0^\pi \left(\frac{d\phi}{dx}\right)^2 dx = \int_0^\pi \sin^2 x\, dx = \frac{1}{2}\int_0^\pi (1 - \cos 2x)\, dx = \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right]_0^\pi = \frac{\pi}{2}$$

$$\int_0^\pi \phi^2\, dx = \int_0^\pi (\cos^2 x - 2\cos x + 1)\, dx = \int_0^\pi \left(\frac{1 + \cos 2x}{2} - 2\cos x + 1\right) dx$$

$$= \left[\frac{3}{2}x + \frac{\sin 2x}{4} - 2\sin x\right]_0^\pi = \frac{3}{2}\pi$$

$$\therefore \lambda^2 = \left[\frac{\frac{\pi}{2}}{\frac{3\pi}{2}}\right] = \frac{1}{3} \quad : \text{ the smallest or first eigenvalue.}$$