MAE 105 - Problem Solving Session 5/25/09

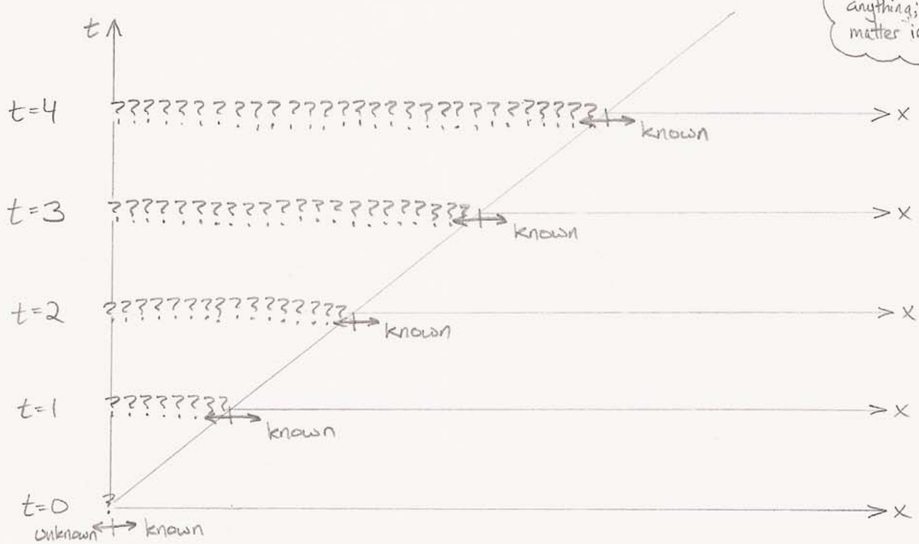
(No office hour or problem session this Monday 5/25, in observance of Memorial Day.)

① Imagine a semi-infinite ($x>0$), thin, uniform string $\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0$
with known initial conditions, $f(x)$ & $g(x)$
but unknown boundary condition, $u(0,t) = ?$

Q: Where and when (x,t) can you still determine what the string is doing (i.e. determine $u(x,t)$)?

A: Information (or a lack of information) travels with speed $c=1$, so any points below the forward characteristic from $x=0$ are completely defined by the initial conditions and can be determined.

The boundary condition could be anything; it doesn't matter in this region!

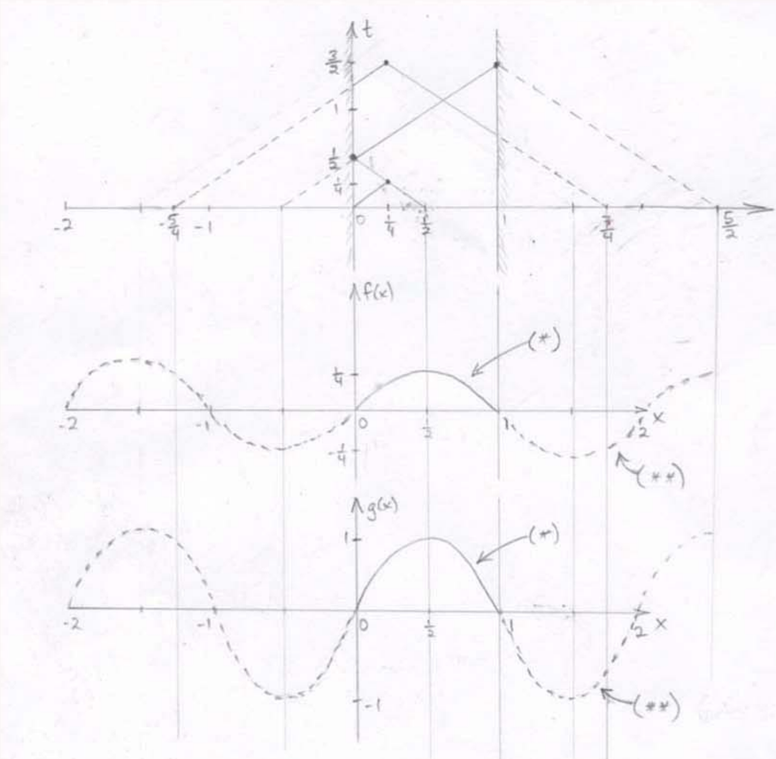


② Consider: $\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0$, $\quad t>0, \quad 0<x<1$

B.C.'s: $u(0,t) = u(1,t) = 0$

I.C.'s: $u(x,0) = x(1-x) = f(x)$

$\frac{\partial u}{\partial x}(x,0) = \sin \pi x = g(x)$



* initial conditions

** extended initial conditions such that the boundary conditions are satisfied for all $t>0$

a) verify boundary conditions are satisfied at $(x=0, t=\frac{1}{2})$, $(x=1, t=\frac{3}{2})$

gen soln: $u(x,t) = \frac{1}{2}\left\{f(x+t)+f(x-t)\right\} + \frac{1}{2}\int_{x-t}^{x+t} g(\bar{x})\,d\bar{x}$

$u(0,\frac{1}{2}) = \frac{1}{2}\left(f(\frac{1}{2})+f(-\frac{1}{2})\right) + \frac{1}{2}\int_{-1/2}^{1/2} \sin \pi x\,dx = \frac{1}{2}\left(\frac{1}{4}-\frac{1}{4}\right) + \cancel{\frac{1}{2}\left[\frac{-1}{\pi}\cos \pi x\right]_{-1/2}^{1/2}} = 0$ ✓

g(x) extends as $\sin \pi x$ for all x

$u(1,\frac{3}{2}) = \frac{1}{2}\left(f(\frac{5}{2})+f(-\frac{1}{2})\right) + \frac{1}{2}\int_{-1/2}^{5/2} \sin \pi x\,dx = \frac{1}{2}\left(\frac{1}{4}-\frac{1}{4}\right) + \cancel{\frac{1}{2}\left[-\frac{1}{\pi}\cos \pi x\right]_{-1/2}^{5/2}} = 0$ ✓

b) find values at $\left(x=\frac{1}{4}, x=\frac{1}{4}\right)$, $\left(x=\frac{1}{4}, x=\frac{3}{2}\right)$

$$u\left(\tfrac{1}{4},\tfrac{1}{4}\right) = \tfrac{1}{2}\left(f\left(\tfrac{1}{2}\right)+f(0)\right) + \tfrac{1}{2}\int_0^{1/2} \sin\pi x\,dx$$

$$= \tfrac{1}{2}\left(\tfrac{1}{4}+0\right) + \tfrac{1}{2}\left[-\tfrac{1}{\pi}\cos\pi x\right]_0^{1/2}$$

$$= \boxed{\tfrac{1}{8}+\tfrac{1}{2\pi}}$$

g(x) extends as $\sin\pi x$ for all x
(be careful; might not always have g(x) where this is the case → integrate piecewise)

$$u\left(\tfrac{1}{4},\tfrac{3}{2}\right) = \tfrac{1}{2}\left(f\left(\tfrac{7}{4}\right)+f\left(-\tfrac{5}{4}\right)\right) + \tfrac{1}{2}\int_{-5/4}^{7/4} \sin\pi x\,dx$$

$f(x) = (x-2)(1+(x-2))$

$f(x) = (x+2)(1-(x+2))$

$$= \tfrac{1}{2}\left(-\tfrac{3}{16}+\tfrac{3}{16}\right) + \tfrac{1}{2}\left[-\tfrac{1}{\pi}\cos\pi x\right]_{-5/4}^{7/4}$$

$$= \boxed{-\tfrac{\sqrt{2}}{2\pi}}$$