MAE105 - Problem Solving Session

5/11/09

① Consider the following ODE: $y'' + \alpha^2 y = 0$

and a series solution of the form $y(x) = \sum_{n=0}^{\infty} a_n x^n$

a) Substitute the infinite series into the ODE and find the first six a_n's (i.e. a_0 through a_5). Write down the series.

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)'' + \alpha^2\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0$$

$$\sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} + \sum_{n=0}^{\infty} a_n \alpha^2 x^n = 0$$

$$\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + \sum_{n=2}^{\infty} a_{n-2}\alpha^2 x^{n-2} = 0$$

$$\sum_{n=2}^{\infty} \underbrace{\left(a_n n(n-1) + a_{n-2}\alpha^2\right)}_{=0} \underbrace{x^{n-2}}_{\neq 0} = 0$$

for $n=2$: $a_2 2(2-1) + a_0 \alpha^2 = 0$

$$a_2 = \frac{-\alpha^2 a_0}{2}$$

for $n=3$: $a_3 3(3-1) + a_1 \alpha^2 = 0$

$$a_3 = -\frac{\alpha^2 a_1}{6}$$

for $n=4$: $a_4 4(4-1) + a_2 \alpha^2 = 0$

$$a_4 = -\frac{\alpha^2 a_2}{12} = \frac{\alpha^4 a_0}{24}$$

for $n=5$: $a_5 5(5-1) + a_3 \alpha^2 = 0$

$$a_5 = -\frac{\alpha^2 a_3}{20} = \frac{\alpha^4 a_1}{120}$$

$$\therefore \boxed{y(x) = a_0 + a_1 x - \frac{\alpha^2 a_0}{2}x^2 - \frac{\alpha^2 a_1}{6}x^3 + \frac{\alpha^4 a_0}{24}x^4 + \frac{\alpha^4 \alpha_1}{120}x^5 - \dots}$$

b) Directly solve the ODE and find a closed form solution.

$$\boxed{y(x) = A\cos(\alpha x) + B\sin(\alpha x)}$$

c) Are the two solutions the same?

Group even & odd terms in solution (a):

$$y(x) = \underbrace{\left[a_0 - \frac{\alpha^2 a_0}{2}x^2 + \frac{\alpha^4 a_0}{24}x^4 - \dots\right]}_{\text{even}} + \underbrace{\left[a_1 x - \frac{\alpha^2 a_1}{6}x^3 + \frac{\alpha^4 a_1}{120}x^5 - \dots\right]}_{\text{odd}}$$

$$= a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(\alpha x)^{2n} + \frac{a_1}{\alpha} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(\alpha x)^{2n+1}$$

$$= a_0 \cos(\alpha x) + \bar{a}_1 \sin(\alpha x)$$

$\therefore$ $\boxed{\text{Yes,}}$ the two solutions are the same.

② a) Find a general expression $x = x(t)$, for the characteristics of the following PDE:

$$\frac{\partial u}{\partial t} - x\cos t \frac{\partial u}{\partial x} = -u\sin t \qquad (\text{where } x(0) = x_0)$$

Characteristic $\equiv$ $x(t)$ for which the total time derivative of u $\left(\frac{du}{dt}\right)$ is equal to the RHS

For this problem, want $\dfrac{du}{dt} = -u\sin t$

So, to rewrite, we want $\dfrac{du}{dt} = \dfrac{\partial u}{\partial t} - x\cos t \dfrac{\partial u}{\partial x}$

Since $u = u(x(t), t)$, by the chain rule, $\dfrac{du}{dt} = \dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial x}\dfrac{dx}{dt}$

Therefore, $\dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial x}\dfrac{dx}{dt} = \dfrac{\partial u}{\partial t} - x\cos t \dfrac{\partial u}{\partial x}$

$$\frac{dx}{dt} = -x\cos t$$

$$\int \frac{dx}{x} = \int -\cos t\, dt$$

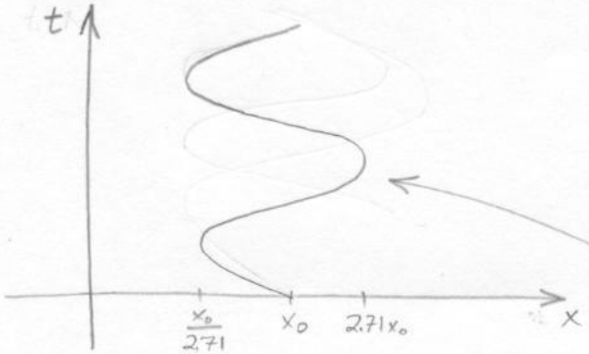
$$\ln x = -\sin t + C$$

$$x = Ae^{-\sin t}$$

$$x(0) = Ae^0 = x_0$$

$$\boxed{x = x_0 e^{-\sin t}}$$

b) In the x,t-plane, $t>0$, sketch a typical characteristic curve



Range: $x_0 e^{-1} < x < x_0 e^{1}$

$\frac{x_0}{2.71} < x < 2.71 x_0$.

Slope @ $t=0$: $x'(0) = x_0(-\cos 0)e^{0}$

$= -x_0$

$\frac{du}{dt} = -u\sin t$ on this line

c) Find the general solution of this PDE.

$$\frac{du}{dt} = -u\sin t$$

$$\frac{du}{u} = -\sin t\,dt$$

$$\ln u = \cos t + C$$

$$\boxed{u = \bar{C}e^{\cos t}}$$

d) Specialize the solution in (c) such that at $t=0$, we have

$$u(x_0,0) = u_0 = (x_0^2+1)$$

$$u(x_0,0) = \bar{C}e^{1} = (x_0^2+1)$$

$$\bar{C} = \frac{x_0^2+1}{e}$$

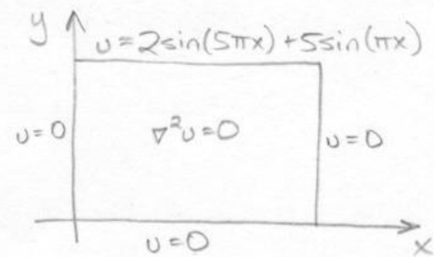
$$u(x,t) = \frac{x_0^2+1}{e}e^{\cos t} \quad \text{where } x_0 = xe^{\sin t}$$

$$= \boxed{\frac{x^2e^{2\sin t}+1}{e}e^{\cos t}}$$

③ u is harmonic in a $1\times H$ plate: $\nabla^2 u = 0$, $0<x<1$, $0<y<H$

B.C.'s: $u(x,0)=0$ $\quad u(x,H) = 2\sin(5\pi x) + 5\sin(\pi x)$

$u(0,y)=0$ $\quad u(1,y)=0$



a) Separation of variables: $u(x,y) = \phi(x) G(y)$

$$G \frac{d^2\phi}{dx^2} + \phi \frac{d^2 G}{dt^2} = 0$$

$$\frac{1}{\phi}\frac{d^2\phi}{dx^2} = -\frac{1}{G}\frac{d^2G}{dt^2} = \underbrace{-\lambda, \quad \lambda \geq 0}_{\text{makes } \phi(x) \text{ periodic}}$$

b) $\frac{d^2\phi}{dx^2} + \lambda\phi = 0$ and $\frac{d^2G}{dy^2} - \lambda G = 0$

$\phi(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$

$G(y) = C\cosh(\sqrt{\lambda}x) + D\sinh(\sqrt{\lambda}y)$

c) apply B.C.'s: $u(x,0) = 0 \rightarrow G(0) = 0$

$u(0,y) = 0 \rightarrow \phi(0) = 0$

$u(1,y) = 0 \rightarrow \phi(1) = 0$

d) $\phi(0) = A\cos 0 + \cancel{B\sin 0} = 0$

$A = 0$

$\phi(1) = B\sin\sqrt{\lambda} = 0$

$\lambda_n = (n\pi)^2$ where $n = 1, 2, \ldots$

$\phi_n(x) = B_n \sin(n\pi x)$

e) $G(0) = C\cosh 0 + \cancel{D\sinh 0} = 0$

$C = 0$

$G_n(y) = D_n \sinh(n\pi y)$

$$u(x,y) = \sum_{n=1}^{\infty} E_n \sin(n\pi x)\sinh(n\pi y)$$

f) apply last B.C.: $u(x,H) = 2\sin(5\pi x) + 5\sin(\pi x) = \sum_{n=1}^{\infty} E_n \sin(n\pi x)\sinh(n\pi H)$

$$\left.\begin{array}{l} \text{for } n=1 \rightarrow E_1 = \dfrac{5}{\sinh(\pi H)} \\ \text{for } n=5 \rightarrow E_5 = \dfrac{2}{\sinh(5\pi H)} \end{array}\right\} \text{by orthogonality, } E_n = 0 \text{ for all other } n$$

$$\boxed{u(x,y) = \frac{2}{\sinh(5\pi H)}\sin(5\pi x)\sinh(5\pi y) + \frac{5}{\sinh(\pi H)}\sin(\pi x)\sinh(\pi y)}$$

Good Luck on Midterm #2