MAE105 - Problem Solving Session

5/4/09

① Say we are given:

$$H(x)\frac{d^2\phi}{dx^2} + \frac{H(x)}{1-\cos x}\frac{d\phi}{dx} + H(x)\gamma(x)\phi + \lambda H(x)\tilde{\sigma}(x)\phi = 0 \qquad (1)$$

We need to determine $H(x)$ such that (1) is a Sturm-Liouville DE.

→ Recall the Sturm-Liouville Differential Equation:

$$\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\phi + \lambda\sigma(x)\phi = 0$$

$$p(x)\frac{d^2\phi}{dx^2} + \frac{dp(x)}{dx}\frac{d\phi}{dx} + q(x)\phi + \lambda\sigma(x)\phi = 0 \qquad (2)$$

→ For (1) = (2), equate the coefficients:

$$H(x) = p(x) \qquad *$$

$$\frac{H(x)}{1-\cos x} = \frac{dp(x)}{dx} \qquad **$$

$$H(x)\gamma(x) = q(x)$$

$$H(x)\tilde{\sigma}(x) = \sigma(x)$$

→ From * and **:

$$\frac{dH}{dx} = \frac{H}{1-\cos x}$$

$$\int\frac{dH}{H} = \int\frac{dx}{1-\cos x}$$

$$\ln H = \int\frac{1}{2\sin^2\frac{x}{2}}dx \qquad \text{since } \sin^2 x = \frac{1-\cos 2x}{2}$$

$$\ln H = -\cot\frac{x}{2} + C$$

$$H(x) = e^{-\cot\frac{x}{2}+C} = e^C e^{-\cot\frac{x}{2}} = \boxed{Ae^{-\cot\frac{x}{2}}}$$

Now, suppose we are given a little more information and (1) is rewritten as:

$$\frac{d}{dx}\left(Ae^{-\cot\frac{x}{2}}\frac{d\phi}{dx}\right) + \lambda\phi = 0 \quad \text{for } 0<x<\pi \qquad (3)$$

with B.C.'s: $\phi(0) = \phi(\pi) = 0$

and: $\phi_t = \tan x$

We need to estimate the first eigenvalue.

(= use the Rayleigh quotient)

→ Observe in (3): $\sigma(x) = 1, \quad q(x) = 0$

→ RQ: $\lambda = \dfrac{-p\phi \frac{d\phi}{dx}\Big|_a^b + \int_a^b \left[p\left(\frac{d\phi}{dx}\right)^2 - q\phi^2\right]dx}{\int_a^b \phi^2 \sigma \, dx}$

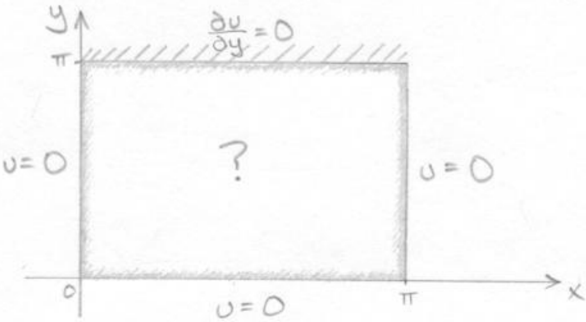
$$\lambda_1 = \frac{-Ae^{-\cot\frac{x}{2}}\cancel{\tan x}\cdot\frac{1}{\cos^2 x}\Big|_0^\pi + \int_0^\pi \left[Ae^{-\cot\frac{x}{2}}\cdot\frac{1}{\cos^4 x} - 0\right]dx}{\int_0^\pi \tan^2 x \, dx}$$

$\lambda_1 = \ldots$ (complete integrations)

② Temperature in a rectangular region: $u \equiv u(x,y,t)$

$$\frac{\partial u}{\partial t} - \nabla^2 u = 0, \quad 0 < x < \pi, \quad 0 < y < \pi, \quad t > 0$$

Boundary conditions: $u(0,y,t) = u(\pi,y,t) = u(x,0,t) = \frac{\partial u}{\partial y}(x,\pi,t) = 0$



→ Separation of variables #1: $u(x,y,t) = h(t)\phi(x,y)$

$$\frac{\partial u}{\partial t} - \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0$$

$$\phi\frac{dh}{dt} - h\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) = 0$$

$$\frac{1}{h}\frac{dh}{dt} = \frac{1}{\phi}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) = -\lambda \qquad \lambda > 0$$

$$\frac{dh}{dt} + \lambda h = 0$$

$$\underline{h(t) = Ae^{-\lambda t}}$$

Separation of variables #2: $\phi(x,y) = f(x)g(y)$

$$g\frac{d^2 f}{dx^2} + f\frac{d^2 g}{dy^2} + \lambda f g = 0$$

$$\frac{1}{f}\frac{d^2 f}{dx^2} = -\frac{1}{g}\frac{d^2 g}{dy^2} - \lambda = -\mu \qquad \mu > 0$$

$$\frac{d^2 f}{dx^2} + \mu f = 0$$

$$\underline{f(x) = B\cos\sqrt{\mu}x + C\sin\sqrt{\mu}x}$$

$$\frac{d^2 g}{dy^2} + \lambda g = \mu g$$

$$\frac{d^2 g}{dy^2} + (\lambda - \mu)g = 0$$

$$\underline{g(y) = D\cos\sqrt{\lambda-\mu}\,y + E\sin\sqrt{\lambda-\mu}\,y}$$

→ Boundary conditions: $u(0,y,t) = f(0)g(y)h(t) = 0 \longrightarrow f(0) = 0$

$u(\pi,y,t) = f(\pi)g(y)h(t) = 0 \longrightarrow f(\pi) = 0$

$u(x,0,t) = f(x)g(0)h(t) = 0 \longrightarrow g(0) = 0$

$\frac{\partial}{\partial y}u(x,\pi,t) = f(x)g'(\pi)h(t) = 0 \longrightarrow g'(\pi) = 0$

→ Apply B.C.'s: $g(y) \rightarrow g(0) = D\cos 0 + E\cancel{\sin 0} = 0$

$$D = 0$$

$$g'(\pi) = E\sqrt{\lambda-\mu}\cos\sqrt{\lambda-\mu}\,\pi = 0$$

Since $\sqrt{\lambda-\mu} \neq 0$ and $E \neq 0$,

$$\cos\sqrt{\lambda-\mu}\,\pi = 0$$

$$\sqrt{\lambda-\mu}\,\cancel{\pi} = \left(\frac{2n+1}{2}\right)\cancel{\pi} \quad \text{for } n = 0,1,2,\ldots$$

$$\lambda-\mu = \left(\frac{2n+1}{2}\right)^2 \qquad (*)$$

$$g_n(y) = E_n \sin\left(\frac{2n+1}{2}y\right)$$

$f(x) \rightarrow f(0) = B\cos 0 + C\cancel{\sin 0} = 0$

$$B = 0$$

$$f(\pi) = C\sin\sqrt{\mu}\,\pi = 0$$

$$\sqrt{\mu}\,\cancel{\pi} = m\cancel{\pi} \quad \text{for } m = 1,2,\ldots$$

$$\mu = m^2$$

$$f_m(x) = C_m \sin mx$$

→ Determine λ from $(*)$: $\lambda_{mn} - \mu_m = \left(\frac{2n+1}{2}\right)^2$

$$\lambda_{mn} = \left(\frac{2n+1}{2}\right)^2 + m^2$$

→ General Solution: $u(x,y,t) = f(x)g(y)h(t)$

$$= \boxed{\sum_{m=1}^{\infty}\sum_{n=0}^{\infty} F_{mn}\sin(mx)\sin\left(\frac{2n+1}{2}y\right)e^{-\left(\left(\frac{2n+1}{2}\right)^2+m^2\right)t}}$$

(Now can use initial condition and orthogonality to find F_{mn})