

Name: _____

Time: 3:35 to 4:50pm

Date: April 23, 2009

MAE105
First Midterm Exam
(open book, closed notes)

Problem 1:

Consider the following differential equation:

$$\frac{d^2u}{dx^2} - 8x^3 = 0, \quad 0 < x < 2, \quad (4)$$

with the following boundary conditions:

$$u(0) = 1, \quad u(2) = 8. \quad (5)$$

- (a) (2.5 Points) Integrate the equation twice to obtain an explicit expression for $u(x)$ which must include two integration constants.
- (b) (2.5 Points) Use the boundary conditions to find the constants of integration and write down explicitly the final solution.

Problem 2:

Consider the following diffusion PDE:

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0, \quad t > 0, \quad 0 < x < \pi, \quad (1)$$

with the following boundary conditions:

$$\frac{\partial u}{\partial x}(0, t) = 0, \quad \frac{\partial u}{\partial x}(\pi, t) = 0. \quad (2)$$

- (a) (0.5 Point) Based on the method of "separation of variables", set $u(x, t) = \phi(x)G(t)$, differentiate as necessary, and substitute into the PDE (1).
- (b) (1 Point) Collect terms such that a function of only x is set equal to a function of only t , and use this fact to find two ODE's, one for $\phi(x)$, and the other for $G(t)$, in such a way that the solution for $\phi(x)$ would be periodic.
- (c) (2 Point) Write down the general solution of the ODE for $\phi(x)$. Is your solution periodic?
- (d) (1 Point) Write down the general solution of the ODE for $G(t)$. Does your solution decay exponentially in time?
- (e) (3 Points) Use the boundary conditions (2) to obtain the values of $\phi'(x)$ at $x = 0$ and $x = \pi$. Use the values of $\phi'(x)$ at $x = 0$ and $x = \pi$ to find the eigenvalues and the corresponding eigenfunctions of this problem.
- (i) Is the constant function an eigenfunction?
- (ii) Show by direct calculation that the eigenfunctions (including the constant function) are in fact orthogonal.
- (f) (2.5 Points) Consider the following initial condition:

$$u(x, 0) = f(x) = 2 + 3x, \quad 0 < x < \pi. \quad (3)$$

Expand $f(x)$ in the cosine Fourier series, *i.e.*, in $\phi_n = \cos nx$, $n = 0, 1, 2, \dots$, calculate all the Fourier coefficients to obtain explicit expressions for them, and write down the infinite series representation of the $f(x)$ explicitly.

(i) Is this Fourier representation of $f(x)$ an odd or an even extension of x (which was *originally* defined for $0 < x < \pi$ only)?

(ii) Is this series periodic in x ?

(iii) What is the period if your answer is yes in (ii) above?

(g) (1 Point) Based on the above results, write down the series solution (with explicit coefficients) of the initial-boundary-value problem defined by expressions (1), (2), and (3).

Make sure to write your name at the top of each page.

Good luck!