Sturm-Liouville Eigenvalue Problems

$$\frac{d}{dx}\left(p\frac{d\phi}{dx}\right)+q\phi+\lambda\sigma\phi=0$$

$$a<x<b$$

Boundary Conditions:

$$\beta_1\phi(a)+\beta_2\frac{d\phi}{dx}(a)=0$$

$$\beta_3\phi(b)+\beta_4\frac{d\phi}{dx}(b)=0$$

Simple case: $p \quad 1, q \quad 0, \qquad 1$

$$\frac{d^2\phi(x)}{dx^2}+\lambda\phi(x)=0$$

Heat flow in a nonuniform rod: $p = K_0, q = \alpha, \sigma = c\rho$

$$\frac{d}{dx}\left(K_0 \frac{d\phi}{dx} \right) + \alpha\phi + \lambda c\rho\phi = 0$$

Vibrating of a nonuniform string:

$$p = T_0, q = \alpha, \sigma = \rho_0$$

$$T_0 \frac{d^2\phi}{dx^2} + \alpha\phi + \lambda\rho_0\phi = 0$$

Sturm-Liouville Eigenvalue Problems

$$\frac{d}{dx}\left(p(x)\frac{d\phi(x)}{dx}\right)+q(x)\phi+\lambda\sigma(x)\phi=0 \qquad a<x<b$$

With B.C's

1. **All the eigenvalues λ are real.**
2. **Inifinite number of eigenvalues**

$$\lambda_1<\lambda_2<\lambda_3<\ldots<\lambda_n$$

$$\lambda_n\to\infty \quad \text{as} \quad n\to\infty$$

3. $\phi_n(x)$ **has n-1 zeros.**

4. The eigenfunctions form a "complete" set:

$$f(x) \sim \sum_{n=1}^{\infty} a_n \phi_n(x)$$

5. Eigenfunctions are orthogonal:

$$\int_a^b \phi_n(x)\phi_m(x)\sigma(x)dx = 0 \qquad \lambda_n \neq \lambda_m$$

6. Each eigenfunction, , and its eigenvalue, , satisfy the Rayleigh quotient:

$$\lambda = \frac{-p\phi \frac{d\phi}{dx}\Big|_a^b + \int_a^b \left[p\left(\frac{d\phi}{dx}\right)^2 - q\phi^2 \right] dx}{\int_a^b \phi^2 \sigma dx}$$

Self-Adjoint Operators and Sturm-Lioville Eigenvalue Problems

$$\frac{d}{dx}\left(p(x)\frac{d\phi(x)}{dx}\right)+q(x)\phi+\lambda\sigma(x)\phi=0$$

$$\beta_1\phi(a)+\beta_2\frac{d\phi}{dx}(a)=0$$

$$\beta_3\phi(b)+\beta_4\frac{d\phi}{dx}(b)=0$$

Linear operator

$$L(y) \equiv \frac{d}{dx}\left(p(x)\frac{dy}{dx} \right) + q(x)\,y$$

Lagrange's identity

$$L(u) \equiv \frac{d}{dx}\left(p(x)\frac{du}{dx} \right) + q(x)\,u$$

$$L(v) \equiv \frac{d}{dx}\left(p(x)\frac{dv}{dx} \right) + q(x)\,v$$

$$uL(v) - vL(u) \equiv \frac{d}{dx}\left[p\left(u\frac{dv}{dx} - v\frac{du}{dx} \right) \right]$$

Green's formula

$$\int_a^b uL(v) - vL(u)dx = p\left(u\frac{dv}{dx} - v\frac{du}{dx} \right)\Bigg|_a^b$$

Self-adjointness

$$p\left(u\frac{dv}{dx} - v\frac{du}{dx} \right)\Bigg|_a^b = 0$$

If u and v are any two functions satisfying the same set of homogeneous boundary conditions (of the same regular Sturm-Liouville type), then

$$\int_a^b uL(v) - vL(u)dx = 0$$

Vibrating Membrane

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \quad t > 0, \quad 0 < x < L, \quad 0 < y < H$$

$$u(0, y, t) = 0 \qquad u(x, 0, t) = 0$$

$$u(L, y, t) = 0 \qquad u(x, H, t) = 0$$

$$u(x, y, 0) = \alpha(x, y)$$

$$\frac{du}{dt}(x, y, 0) = \beta(x, y)$$

$$u(x,y,t) = h(t)\phi(x,y)$$

Substitute into PDE to Obtain:

$$\frac{1}{c^2}\frac{1}{h(t)}\frac{dh(t)}{dt} = \frac{1}{\phi}\left(\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2}\right) = -\lambda$$

$$\frac{d^2 h(t)}{dt^2} = -\lambda c^2 h(t) \qquad \frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} = -\lambda\phi$$

λ : Eigenvalue

$$\phi(0, y) = 0 \qquad \phi(x, 0) = 0$$

$$\phi(L, y) = 0 \qquad \phi(x, H) = 0$$

$$\phi(x, y) = f(x)g(y)$$

$$u(x, y, t) = h(t)f(x)g(y)$$

$$\frac{1}{f}\frac{d^2 f}{dx^2} = -\lambda - \frac{1}{g}\frac{d^2 g}{dy^2} = -\mu$$

$$\frac{d^2 f}{dx^2} = -\mu f$$

$$\frac{d^2 g}{dy^2} = -(\lambda - \mu) g$$

$$\frac{d^2 h(t)}{dt^2} = -\lambda c^2 h(t)$$

$$f(0) = 0 \qquad g(0) = 0$$

$$f(L) = 0 \qquad g(H) = 0$$

$$\frac{d^2 f}{dx^2} = -\mu f$$

$$f(0) = 0$$

$$f(L) = 0$$

$$f(x) = c_1 \cos\sqrt{\mu}\, x + c_2 \sin\sqrt{\mu}\, x,$$

$$f(0) = f(L) = 0 \xrightleftharpoons{\text{Boundary Conditions}}$$

$$f(x) = c_2 \sin(\sqrt{\mu}\, x),$$

$$f(L) = c_2 \sin(\sqrt{\mu}\, L) = 0,$$

$$c_2 \neq 0 \rightarrow \sin(\sqrt{\mu}\, L) = 0,$$

$$\mu_n = \frac{(n\pi)^2}{L^2}, \quad n = 1, 2, \ldots$$

$$\frac{d^2 g_n}{dy^2} = -(\lambda - \mu_n) g_n$$

$$g_n(0) = 0$$

$$g_n(H) = 0$$

$$g_{nm}(y) = c_2 \sin(\frac{m\pi y}{H}),$$

$$\lambda_{nm} = \mu_n + \frac{(m\pi)^2}{H^2} = \frac{(n\pi)^2}{L^2} + \frac{(m\pi)^2}{H^2}, \quad m = 1, 2, \ldots$$

$$\phi_{nm}(x, y) = \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{H}\right)$$

$$n = 1, 2, 3, \ldots$$

$$m = 1, 2, 3, \ldots$$

$$u(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{nm}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)\cos\left(c\sqrt{\lambda_{nm}}\,t\right)$$
$$+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}B_{nm}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)\sin\left(c\sqrt{\lambda_{nm}}\,t\right)$$

$$\alpha(x,y)=\sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty}A_{nm}\sin\left(\frac{n\pi x}{L}\right)\right)\sin\left(\frac{m\pi y}{H}\right)$$

$$A_{nm}=\frac{2}{L}\int_0^L\left[\frac{2}{H}\int_0^H\alpha(x,y)\sin\left(\frac{m\pi y}{H}\right)dy\right]\sin\left(\frac{n\pi x}{L}\right)dx$$

$$\beta(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}c\sqrt{\lambda_{nm}}B_{nm}\sin(\frac{n\pi x}{L})\sin(\frac{m\pi y}{H})$$

$$c\sqrt{\lambda_{nm}}B_{nm}=\frac{4}{LH}\int_0^L\int_0^H\beta(x,y)\sin(\frac{m\pi y}{H})\sin(\frac{n\pi x}{L})dydx$$

Vibrating Circular Membrane and Bessel Functions

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u \qquad t > 0$$

$$u(a, \theta, t) = 0$$

$$u(r, \theta, 0) = \alpha(r, \theta)$$

$$\frac{du}{dt}(r, \theta, 0) = \beta(r, \theta)$$

$$u(r,\theta,t) = h(t)\phi(r,\theta)$$

$$\nabla^2\phi + \lambda\phi = 0$$

$$\phi(a,\theta) = 0$$

$$\phi(r,\theta) = f(r)g(\theta)$$

$$u(r,\theta,t) = h(t)f(r)g(\theta)$$

$$\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}$$

$$\frac{d^2h}{dt^2} = -\lambda c^2 h$$

$$r\frac{d}{dr}\left(r\frac{df}{dr}\right) + \left(\lambda r^2 - \mu\right) f = 0 \qquad \frac{d^2g}{d\theta^2} = -\mu g$$

$$r\frac{d}{dr}\left(r\frac{df}{dr}\right)+\left(\lambda r^2-\mu\right)f=0$$

$$f(a)=0$$

$$\left|f(0)\right|<\infty$$

Bessel Functions

$$\mu_m=m^2 \qquad m=0,1,2,\ldots$$

$$r\frac{d}{dr}\left(r\frac{df}{dr}\right)+\left(\lambda r^2-m^2\right)f=0$$

$$z=\sqrt{\lambda r} \qquad z^2\frac{d^2f}{dz^2}+z\frac{df}{dz}+\left(z^2-m^2\right)f=0$$

$$f=c_1J_m(z)+c_2Y_m(z)$$

$$z\to 0$$

$$J_m(z)\sim\begin{cases}1 & m=0\\ \dfrac{1}{2^m m!}z^m & m>0\end{cases}$$

$$Y_m(z)\sim\begin{cases}\dfrac{2}{\pi}\ln z & m=0\\ -\dfrac{2^m(m-1)!}{\pi}z^{-m} & m>0\end{cases}$$

$$J_m(z) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(z/2\right)^{2k+m}}{k!(k+m)!}$$

Now, some worked out examples with animations