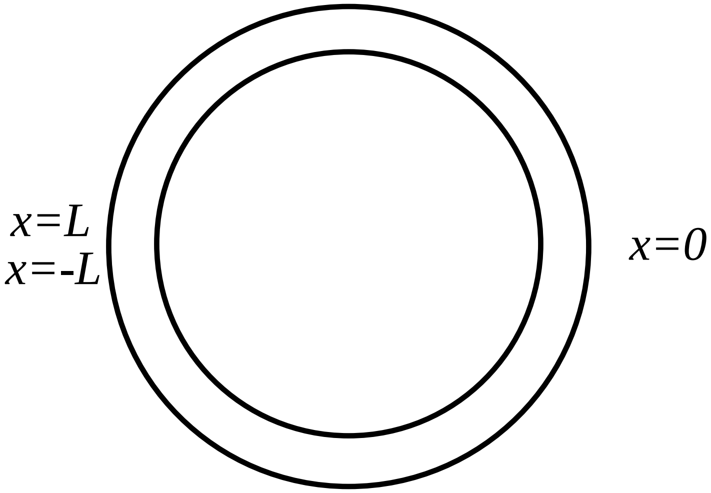
Heat Conduction in a Thin Circular Ring



$$\boxed{\frac{\partial u(x,t)}{\partial t} = k\frac{\partial^2 u(x,t)}{\partial x^2}, \quad k>0, \quad t>0, \quad -L<x<L}$$

$$\text{C.C.}\quad \begin{cases} u(-L,t) = u(L,t) \\ \dfrac{\partial u}{\partial x}(-L,t) = \dfrac{\partial u}{\partial x}(L,t) \end{cases}$$

$$\text{I.C.}\quad \{\; u(x,0) = f(x)$$

Solve by Separation of Variables

$$\boxed{u(x,t)=\phi(x)G(t)}$$

Substitute into PDE to Obtain:

$$\frac{1}{kG(t)}\frac{dG(t)}{dt}=\frac{1}{\phi(x)}\frac{d^2\phi(x)}{dx^2}=-\lambda$$

Function of

time, t, only

Function of

distance, x, only

$$\frac{dG(t)}{dt}=-\lambda kG(t) \qquad \frac{d^2\phi(x)}{dx^2}=-\lambda\phi(x)$$

λ : Eigenvalue

$\phi(x)$: Eigenfunction

Time-dependent Part

$$\boxed{G(t) = ce^{-\lambda k t}}$$

Eigenvalue Problem

$$\boxed{\frac{d^2\phi(x)}{dx^2} + \lambda\phi(x) = 0, \quad plus\ BC's} \qquad \lambda > 0$$

$$\phi(-L) = \phi(L)$$

$$\frac{d\phi}{dx}(-L) = \frac{d\phi}{dx}(L)$$

$$\phi(L) = c_1 \cos(\sqrt{\lambda} L) + c_2 \sin(\sqrt{\lambda} L) = \phi(-L)$$
$$= c_1 \cos(\sqrt{\lambda} L) - c_2 \sin(\sqrt{\lambda} L)$$
$$\sin(\sqrt{\lambda} L) = 0, \rightarrow \lambda_n = \frac{(n\pi)^2}{L^2}, \quad n = 1, 2, \ldots$$

The same eigenvalues are obtained from the other continuity condition:

$$\phi'(x) = -\sqrt{\lambda} c_1 \sin\sqrt{\lambda} x + \sqrt{\lambda} c_2 \cos\sqrt{\lambda} x,$$
$$\phi'(-L) = \phi'(L),$$
$$\sin(\sqrt{\lambda} L) = 0, \rightarrow \lambda_n = \frac{(n\pi)^2}{L^2}, \quad n = 1, 2, \ldots$$

Eigenfunctions **Eigenvalues**

$$\left\{\cos\frac{n\pi x}{L}, \sin\frac{n\pi x}{L}\right\} \longleftrightarrow n \quad 0, 1, 2, 3, \ldots$$

General Solution

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 kt} + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 kt}$$

$$\boxed{f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{L}) + \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{L})}$$

$$\int_{-L}^{L} \cos\frac{n\pi x}{L} \cos\frac{m\pi x}{L}\, dx = \begin{cases} 0 & \mathrm{m} \neq \mathrm{n} \\ \mathrm{L} & \mathrm{m} = \mathrm{n} \neq 0 \\ 2\mathrm{L} & \mathrm{m} = \mathrm{n} = 0 \end{cases}$$

$$\int_{-L}^{L} \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L}\, dx = \begin{cases} 0 & \mathrm{m} \neq \mathrm{n} \\ \mathrm{L} & \mathrm{m} = \mathrm{n} \neq 0 \end{cases}$$

$$\int_{-L}^{L} \cos\frac{n\pi x}{L} \sin\frac{m\pi x}{L}\, dx = 0$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(\frac{n\pi x}{L})dx$$

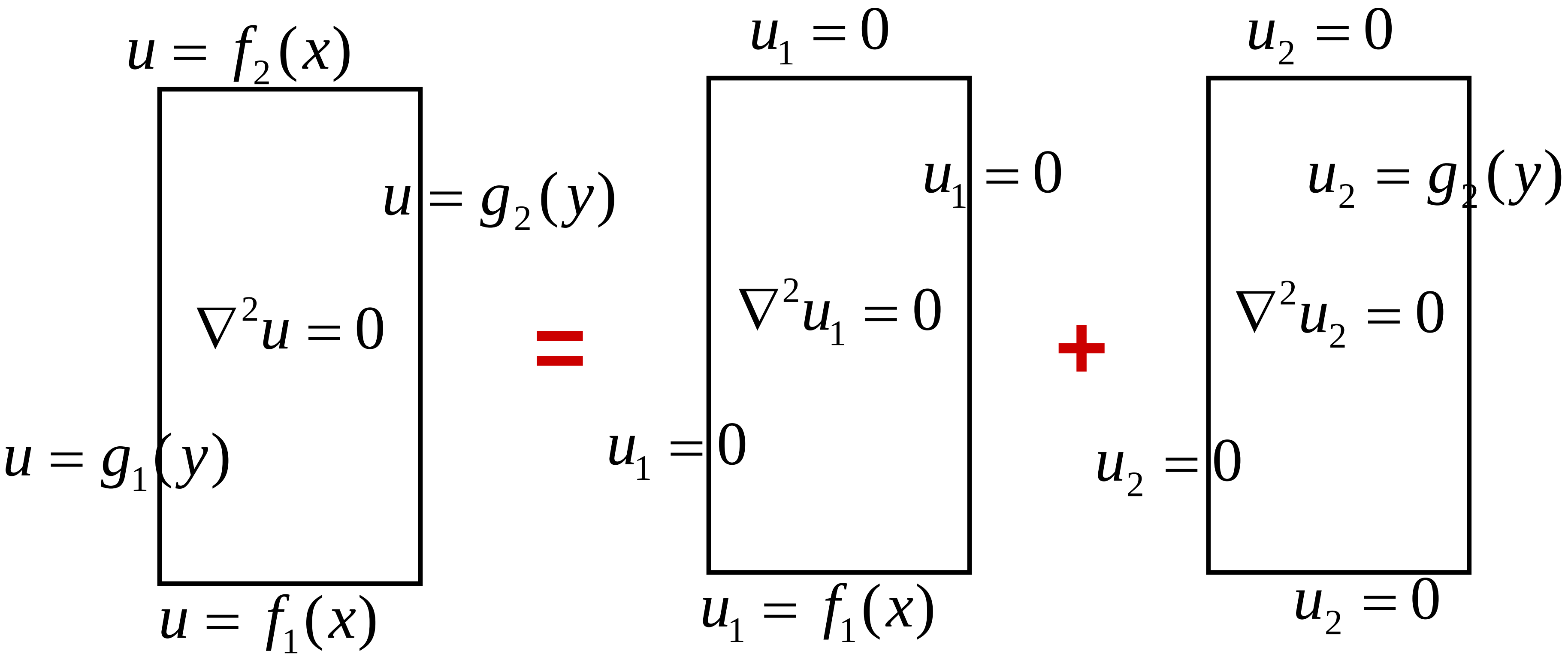
$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(\frac{n\pi x}{L})dx$$

Laplace's Equation Inside a Rectangle

$$\boxed{\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0, \quad 0 < x < L \ , \ 0 < y < H}$$

B.C.'s
$$\begin{cases} u(0,y) = g_1(y) \\ u(L,y) = g_2(y) \\ u(x,0) = f_1(x) \\ u(x,H) = f_2(x) \end{cases}$$

$$u(x,y)=u_1(x,y)+u_2(x,y)+u_3(x,y)+u_4(x,y)$$



$u=f_2(x)$

$u=g_2(y)$

$\nabla^2 u=0$

$u=g_1(y)$

$u=f_1(x)$

=

$u_1=0$

$u_1=0$

$\nabla^2 u_1=0$

$u_1=0$

$u_1=f_1(x)$

+

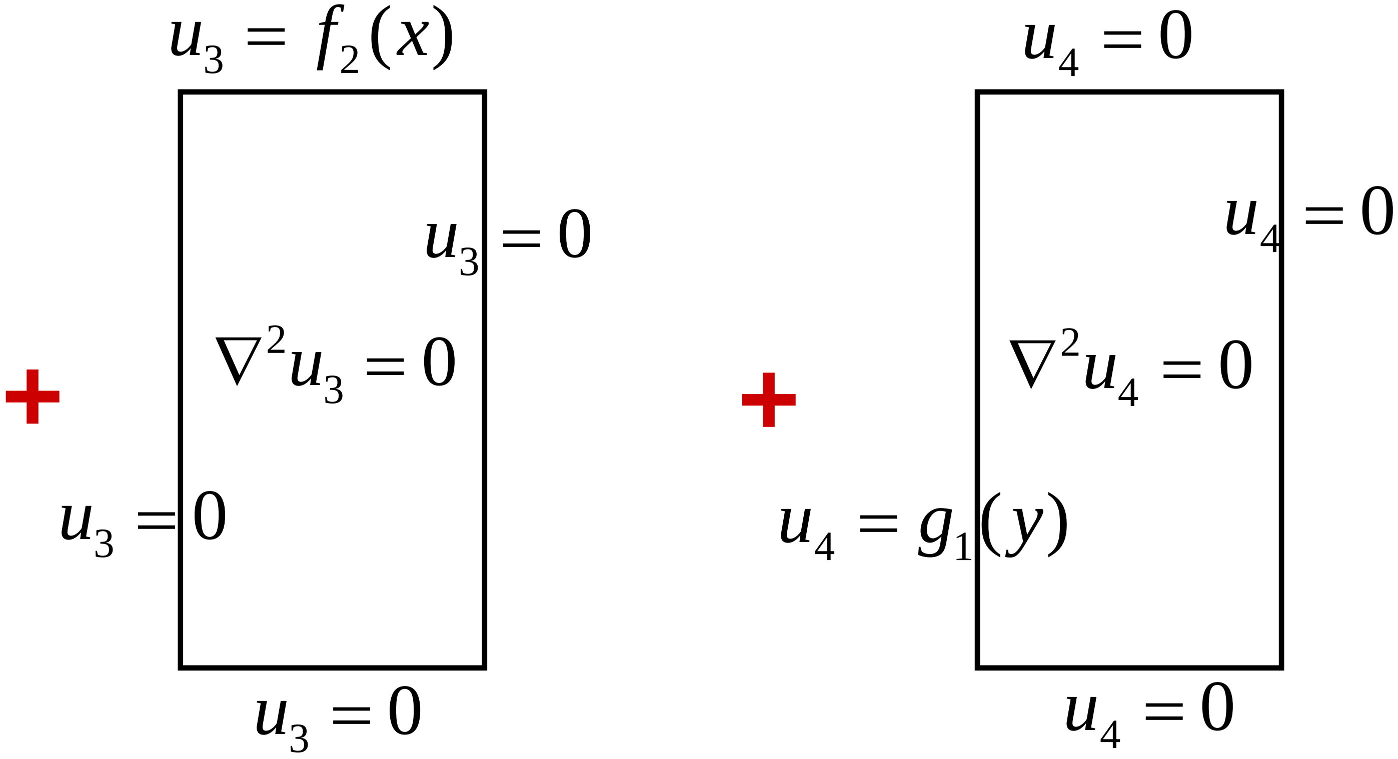
$u_2=0$

$u_2=g_2(y)$

$\nabla^2 u_2=0$

$u_2=0$

$u_2=0$



$+$

$u_3 = f_2(x)$

$u_3 = 0$

$\nabla^2 u_3 = 0$

$u_3 = 0$

$u_3 = 0$

$+$

$u_4 = 0$

$u_4 = 0$

$\nabla^2 u_4 = 0$

$u_4 = g_1(y)$

$u_4 = 0$

$$\boxed{\frac{\partial^2 u_4(x,y)}{\partial x^2}+\frac{\partial^2 u_4(x,y)}{\partial y^2}=0, \quad 0<x<L\ ,\ 0<y<H}$$

$$u_4(0,y)=g_1(y)$$

$$u_4(L,y)=0$$

$$u_4(x,0)=0$$

$$u_4(x,H)=0$$

$$u_4(x,y)=h(x)\phi(y)$$

$$h(L,y)=0$$

$$\phi(0)=0$$

$$\phi(H)=0$$

$$\frac{1}{h(x)}\frac{d^2h(x)}{dx^2} = -\frac{1}{\phi(y)}\frac{d^2\phi(y)}{dy^2} = \lambda$$

Function of x, only Function of y, only

$$\frac{d^2h(x)}{dx^2} = \lambda h(x)$$

$$\frac{d^2\phi(y)}{dy^2} = -\lambda\phi(y)$$

λ : Eigenvalue

ODE

$$\frac{d^2\phi(y)}{dy^2} = -\lambda\phi(y)$$

B.C.'s

$$\phi(0) = 0$$

$$\phi(H) = 0$$

$$\lambda_n = \left(\frac{n\pi}{H}\right)^2, \quad \phi_n(y) = \sin\frac{n\pi}{H}y$$

$$\frac{d^2h(x)}{dx^2} - \lambda h(x) = 0, \quad h(L) = 0$$

$$h(x) = b_0 e^{\sqrt{\lambda}x} + b_1 e^{-\sqrt{\lambda}x}$$

$$h(L) = 0 \longrightarrow b_1 = -b_0 e^{2\sqrt{\lambda}L}$$

$$h(x) = b_0 e^{\sqrt{\lambda}L}(e^{\sqrt{\lambda}(x-L)} - e^{\sqrt{\lambda}(L-x)})$$

$$h(x) = a_2 \sinh[\frac{n\pi}{H}(x-L)]$$

$$h(x) = a_1 \cosh[\frac{n\pi}{H}(x-L)] + a_2 \sinh[\frac{n\pi}{H}(x-L)]$$

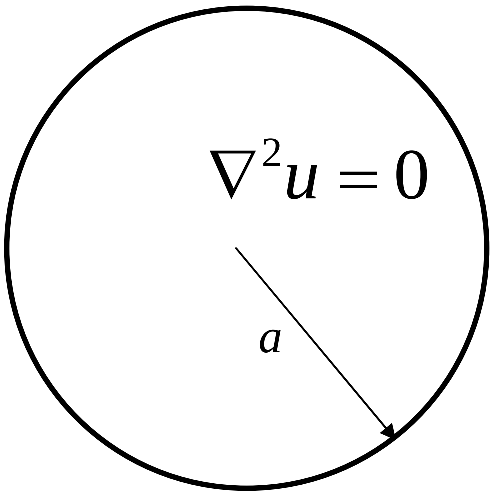
$$h(L) = 0 \quad \longrightarrow \quad h(x) = a_2 \sinh[\frac{n\pi}{H}(x-L)]$$

$$\boxed{u_4(x,t) = \sum_{n=1}^{\infty} a_n \sin(\frac{n\pi y}{H}) \sinh \frac{n\pi}{H}(x-L)}$$

$$\boxed{g_1(y) = \sum_{n=1}^{\infty} a_n \sin(\frac{n\pi y}{H}) \sinh[\frac{n\pi}{H}(-L)]}$$

$$\boxed{a_n = \frac{2}{H \sinh n\pi(-L)/H} \int_0^H g_1(y) \sin(\frac{n\pi y}{H}) dy}$$

Laplace's Equation Inside a Circular Disk



$$0 \le r < a$$

$$-\pi < \theta \le \pi$$

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

$$u(a,\theta) = f(\theta)$$

$$\boxed{\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0}$$

$$u(r,-\pi) = u(r,\pi)$$

$$\left|u(0,\theta)\right| < \infty$$

$$\frac{\partial u}{\partial \theta}(r,-\pi) = \frac{\partial u}{\partial \theta}(r,\pi)$$

$$u(r,\theta) = \phi(\theta)G(r) \longleftarrow$$

Separation of Variables

$$\phi(-\pi) = \phi(\pi) \qquad \frac{d\phi}{d\theta}(-\pi) = \frac{d\phi}{d\theta}(\pi)$$

$$\frac{r}{G(r)}\frac{d}{dr}\left(r\frac{dG(r)}{dr}\right)=-\frac{1}{\phi(\theta)}\frac{d^2\phi(\theta)}{d\theta^2}=\lambda$$

$$\frac{d^2\phi(\theta)}{d\theta^2}=-\lambda\phi(\theta)$$

$$\phi(-\pi)=\phi(\pi) \qquad \frac{d\phi}{d\theta}(-\pi)=\frac{d\phi}{d\theta}(\pi)$$

$$\lambda_n=\left(\frac{n\pi}{L}\right)^2 \quad n=1,2,3,\ldots$$

$$r^2 \frac{d^2G(r)}{dr^2} + r\frac{dG(r)}{dr} - n^2G(r) = 0$$

$$|G(0)| < \infty$$

$$G(r) = c_1 r^n$$

$$\boxed{u(r,\theta) = \sum_{n=1}^{\infty} A_n r^n \cos n\theta + \sum_{n=1}^{\infty} B_n r^n \sin n\theta}$$

$$0 \le r < a$$

$$-\pi < \theta \le \pi$$

$$\boxed{f(\theta) = \sum_{n=0}^{\infty} A_n a^n \cos n\theta + \sum_{n=0}^{\infty} B_n a^n \sin n\theta}$$

$$-\pi < \theta \le \pi$$

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,dx$$

$$A_n a^n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos(n\theta)\,d\theta$$

$$B_n a^n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin(n\theta)\,d\theta$$