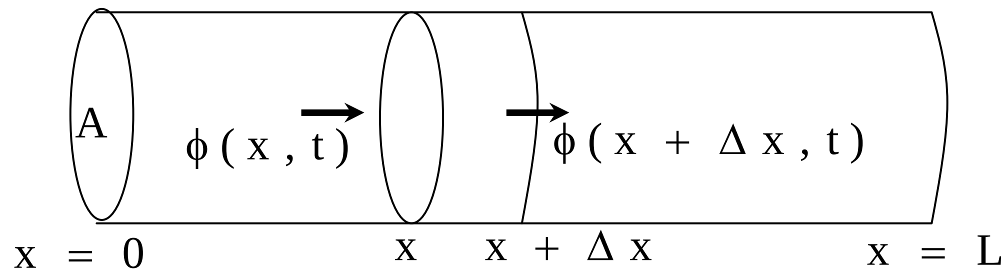
Heat Equation with Zero Temperature at Ends

No heat generation, $Q(x,t)=0$



Partial Differential Equation:
Defines the variation of the dependent variable, *u(x, t)*, inside the domain of variation of independent variables, *t and x*

$$\frac{\partial u(x,t)}{\partial t}=k\frac{\partial^2 u(x,t)}{\partial x^2},\quad t>0,\quad 0<x<L$$

Boundary Conditions: Define the dependent variable, *u(x, t)*, at the end points

$$u(0,t)=0$$

$$u(L,t)=0$$

Initial Condition: Defines the dependent variable, *u(x, t)*, at the initial time

$$u(x,0)=f(x)$$

Solve by Separation of Variables

$$u(x,t) = \phi(x)G(t)$$

Substitute into PDE to Obtain:

$$\frac{1}{kG(t)}\frac{dG(t)}{dt} = \frac{1}{\phi(x)}\frac{d^2\phi(x)}{dx^2} = -\lambda$$

Function of time, t, only

Function of distance, x, only

$$\frac{dG(t)}{dt} = -\lambda k G(t)$$

$$\frac{d^2\phi(x)}{dx^2} = -\lambda\phi(x)$$

λ : Eigenvalue

$\phi(x)$: Eigenfunction

Time-dependent Part

$$G(t) = ce^{-\lambda kt}$$

Eigenvalue Problem

$$\frac{d^2\phi(x)}{dx^2} + \lambda\phi(x) = 0, \quad plus\ BC's$$

Case I : $\lambda > 0$

$$\phi(x) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x,$$

$$\phi(0) = \phi(L) = 0 \xrightleftharpoons{Boundary\ Conditions}$$

$$\phi(x) = c_2 \sin(\sqrt{\lambda}x),$$

$$\phi(L) = c_2 \sin(\sqrt{\lambda}L) = 0,$$

$$c_2 \neq 0 \rightarrow \sin(\sqrt{\lambda}L) = 0,$$

$$\lambda_n = \frac{(n\pi)^2}{L^2}, \quad n = 1, 2, \ldots$$

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$$\lambda_n = n^2\pi^2 / L^2, \quad n = 1, 2, \ldots,$$

$$\phi_n(x) = d_n \sin(\frac{n\pi x}{L})$$

$$u(x,t) = \phi(x) G(t)$$

$$G_n(t) = c_n e^{-(\frac{n\pi}{L})^2 kt}$$

$$u_n(x,t) = \phi_n(x) G_n(t) = c_n d_n e^{-(\frac{n\pi}{L})^2 kt} \sin(\frac{n\pi x}{L})$$

$$= a_n e^{-(\frac{n\pi}{L})^2 kt} \sin(\frac{n\pi x}{L}), \quad n = 1, 2, \ldots$$

Linear & Homogeneous PDE Operator:

$$\frac{\partial^2}{\partial x^2}(c_1 u_1 + c_2 u_2) = c_1 \frac{\partial^2 u_1}{\partial x^2} + c_2 \frac{\partial^2 u_2}{\partial x^2}$$

$$\frac{\partial}{\partial t}(c_1 u_1 + c_2 u_2) = c_1 \frac{\partial u_1}{\partial t} + c_2 \frac{\partial u_2}{\partial t}$$

$$L \equiv \frac{\partial}{\partial t} - k \frac{\partial^2}{\partial x^2} \rightarrow PDE\ Operator$$

$$\boxed{L(c_1 u_1 + c_2 u_2) = c_1 L(u_1) + c_2 L(u_2)}$$

Principle of Superposition

If u_1 and u_2 satisfy a linear *homogeneous* equation, then an arbitrary linear combination of them,

$$c_1 u_1 + c_2 u_2$$

also satisfies the same linear homogeneous equation.

Linear & Homogeneous PDE & Boundary conditions:

$$\frac{\partial u(x,t)}{\partial t} = k \frac{\partial^2 u(x,t)}{\partial x^2},$$

$$u(0,t) = 0, \quad u(L,t) = 0$$

Most General Solution

$$u_n(x,t) = \phi_n(x)G_n(t) = c_n d_n e^{-(\frac{n\pi}{L})^2 kt} \sin(\frac{n\pi x}{L})$$

$$= a_n e^{-(\frac{n\pi}{L})^2 kt} \sin(\frac{n\pi x}{L}), \quad n = 1, 2, \ldots$$

$$u(x,t) = \sum_{n=1}^{\infty} \phi_n(x)G_n(t) = \sum_{n=1}^{\infty} a_n e^{-(\frac{n\pi}{L})^2 kt} \sin(\frac{n\pi x}{L})$$

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} a_n \sin(\frac{n\pi x}{L})$$

Orthogonality

$$\int_0^L A(x)B(x)\,dx = 0$$

A(x) and B(x) are orthogonal over the interval $0 < x < L$

Orthogonality of Fourier Sine & Cosine

$$\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx = \begin{cases} 0 & \mathrm{m} \neq \mathrm{n} \\ \dfrac{L}{2} & \mathrm{m} = \mathrm{n} \end{cases}$$

$$\int_0^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx = \begin{cases} 0 & \mathrm{m} \neq \mathrm{n} \\ \dfrac{L}{2} & \mathrm{m} = \mathrm{n} \end{cases}$$

$$\int_{-L}^L \cos\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx = 0$$

Fourier Sine Representation of *f(x)*

Use Orthogonality

$$f(x) \sim \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L}$$

$$a_n = \frac{\int_0^L f(x) \sin \frac{n\pi x}{L} dx}{\int_0^L \sin^2 \frac{n\pi x}{L} dx} = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$$

$$\frac{d^2\phi(x)}{dx^2} + \lambda\phi(x) = 0$$

Case II : $\lambda = 0$

$$\phi(x) = c_1 + c_2 x$$

Case III : $\lambda < 0$

$$\phi(x) = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}$$

$$\cosh x \equiv \frac{e^x + e^{-x}}{2} \qquad \sinh x \equiv \frac{e^x - e^{-x}}{2}$$

$$\phi(x) = c_3 \sinh(\sqrt{-\lambda}x) + c_4 \cosh(\sqrt{-\lambda}x)$$

Fourier Cosine Representation of *f(x)*

Use Orthogonality

$$f(x) \sim \sum_{n=1}^{\infty} b_n \cos\frac{n\pi x}{L}$$

$$b_n = \frac{\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx}{\int_0^L \cos^2\frac{n\pi x}{L}\,dx} = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$$