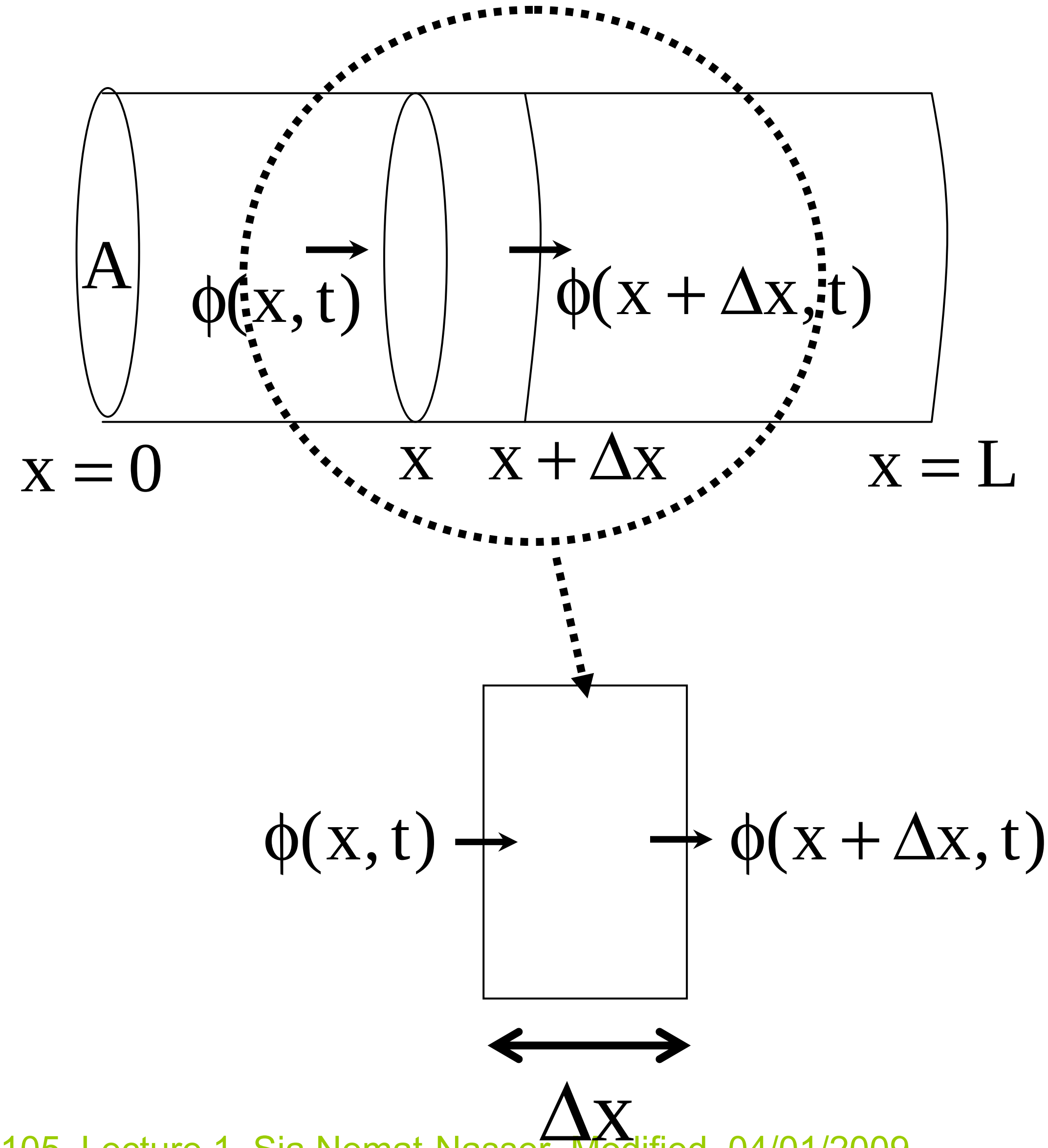
Heat Conduction in a Bar

$\phi(x,t)$ = Heat flux in. It is measured as heat flow per unit area per unit time

$\phi(x+\Delta x,t)$ = Heat flux out

Net rate of heat added to the slice of length Δx is:

$$\left(\phi(x,t)-\phi(x+\Delta x,t)\right)A$$



Conservation of Energy

$$\frac{\partial e(x,t)}{\partial t} = \lim_{\Delta x \to 0} \frac{\phi(x,t) - \phi(x+\Delta x, t)}{\Delta x} = -\frac{\partial \phi}{\partial x}$$

$$e(x,t) = c(x)\rho(x)u(x,t)$$

$e(x, t)$: Internal energy per unit volume

$c(x)$: Heat capacity = Amount of heat necessary to increase the temperature of unit mass by one degree

$\rho(x)$: Mass density = Mass per unit volume

Conservation of Energy

Rate of increase in energy = Rate of energy added

$$c(x)\rho(x)\frac{\partial u}{\partial t} = \lim_{\Delta x \to 0} \frac{\phi(x,t) - \phi(x+\Delta x, t)}{\Delta x} = -\frac{\partial \phi}{\partial x}$$

This is due to heat flux only. If energy is produced internally also, say, due to radiation, at a rate of $Q_0(x,t)$ per unit volume, then we must add this to the right-hand side also.

Set $Q(x,t) = \dfrac{Q_0(x,t)}{c(x)\rho(x)}$ and add to the right-hand

Fourier Law of Conduction

$\phi(x,t)$ = heat flux

Fourier law of conduction:

$$\phi(x,t) = -K(x)\frac{\partial u(x,t)}{\partial x}$$

Heat flows against (positive) temperature gradient, K(x) is conductivity coefficient

<u>Boundary Conditions</u>

If the bar is insulated at x = 0, then

$$\phi(0,t) = 0 \longrightarrow \left.\frac{\partial u(x,t)}{\partial x}\right|_{x=0} = 0$$

One-dimensional Diffusion Equation

$$c(x)\rho(x)\frac{\partial u(x,t)}{\partial t} = -\frac{\partial \phi(x,t)}{\partial x} + Q_0(x,t)$$

$$\phi(x,t) = -K(x)\frac{\partial u(x,t)}{\partial x}$$

$$\boxed{c(x)\rho(x)\frac{\partial u(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left(-K(x)\frac{\partial u(x,t)}{\partial x}\right) + Q_0(x,t)}$$

Homogeneous Material

$$\frac{\partial u(x,t)}{\partial t} = k\frac{\partial^2 u(x,t)}{\partial x^2} + Q(x,t), \quad k = \frac{K}{c\rho} > 0, \quad t > 0, \quad 0 < x < L$$

Boundary conditions:

1. $u(0,t) = 0, \quad u(L,t) = 0$
2. $u_x(0,t) = 0$ insulated, $\quad u_x(L,t) = 0$ insulated
3. $u_x(0,t) = 0$ insulated, $\quad u(L,t) = 0$
4. $u(0,t) = 0, \quad u_x(L,t) = 0$ insulated

These boundary conditions are mutually exclusive

Conservation of Energy

- Rate of change of total energy = Rate of energy added
- Rate of energy added = Net energy flux + Energy generated internally

$$\frac{d}{dt}\int_0^L e(x,t)dx = -\int_0^L \frac{\partial \phi(x,t)}{\partial x}dx + \int_0^L Q(x,t)dx$$
$$= \left[\phi(0,t) - \phi(L,t)\right] + \int_0^L Q(x,t)dx$$

For rate-independent case (equilibrium), we have

$$\left[\phi(L) - \phi(0)\right] = \int_0^L Q(x)dx$$