Heat Equation with General Boundary Conditions

$$\frac{\partial}{\partial t} u = \frac{\partial^2}{\partial x^2} u \qquad 0 < x < \pi$$

Boundary conditions

$$u(0, t) = 0, \; u(\pi, t) + \left(\frac{\partial}{\partial x} u(\pi, t)\right) = 0$$

Initial condition

$$u(x, 0) = f(x)$$

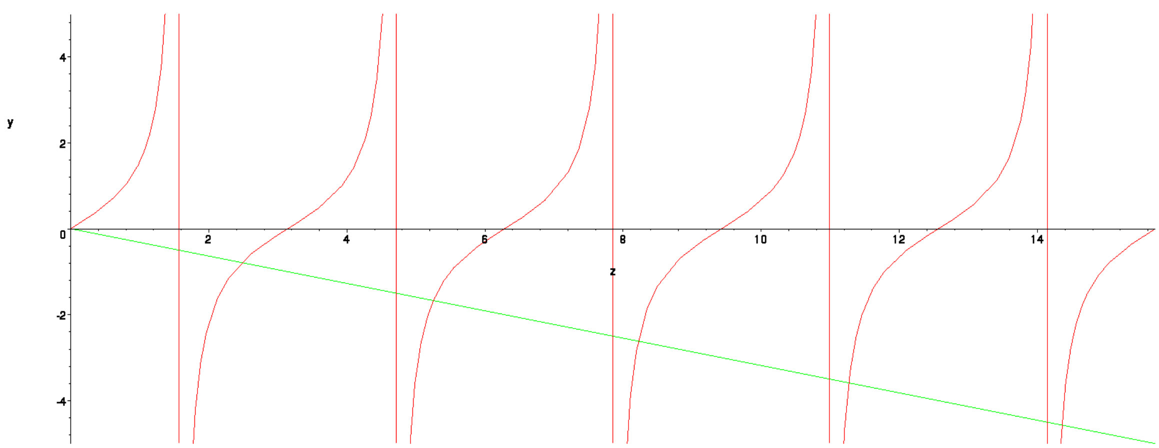
Solution by separation of variables

$$u(x, t) = \phi(x)\, G(t)$$

G(t) = A exp{ $\{-\lambda^2 t\}$

$$\phi(x) = a \sin(\lambda x) + b \cos(\lambda x)$$
$$\phi(0) = 0$$
$$b = 0$$
$$\phi(\pi) + \lambda \left(\frac{\partial}{\partial x} \phi(\pi)\right) = 0$$
$$\sin(\lambda \pi) + \lambda\, \mathrm{coc}(\lambda \pi) = 0$$

Eigenvalues, $\lambda_i, i = 1, 2, \ldots$ are the roots of: $\tan(z) + \frac{z}{\pi} = 0, \; z = \lambda \pi$



Eigenfunctions are:

$$\phi_i(x) = a_i \sin(\lambda_i x), \; i = 1, 2, \ldots$$

We calculate $b_j := \int_0^{\pi} \sin(\lambda_j x)^2 \, dx$

Orthogonality of eigenfunctions:

$$K_{n,m} := \int_0^{\pi} \frac{\sin(\lambda_n x)\sin(\lambda_m x)}{\sqrt{b_n b_m}} \, dx$$

$$-\frac{\lambda_n \cos(\pi \lambda_n)\sin(\pi \lambda_m) - \lambda_m \sin(\pi \lambda_n)\cos(\pi \lambda_m)}{\sqrt{b_n b_m}\,(\lambda_n^2 - \lambda_m^2)}$$ **, for** $n \neq m$,

$$\frac{1}{2}\frac{(-\cos(\pi \lambda_n)\sin(\pi \lambda_n) + \pi \lambda_n)\,\mathrm{csgn}(b_n)}{b_n \lambda_n}$$ **, for** $n = m$.

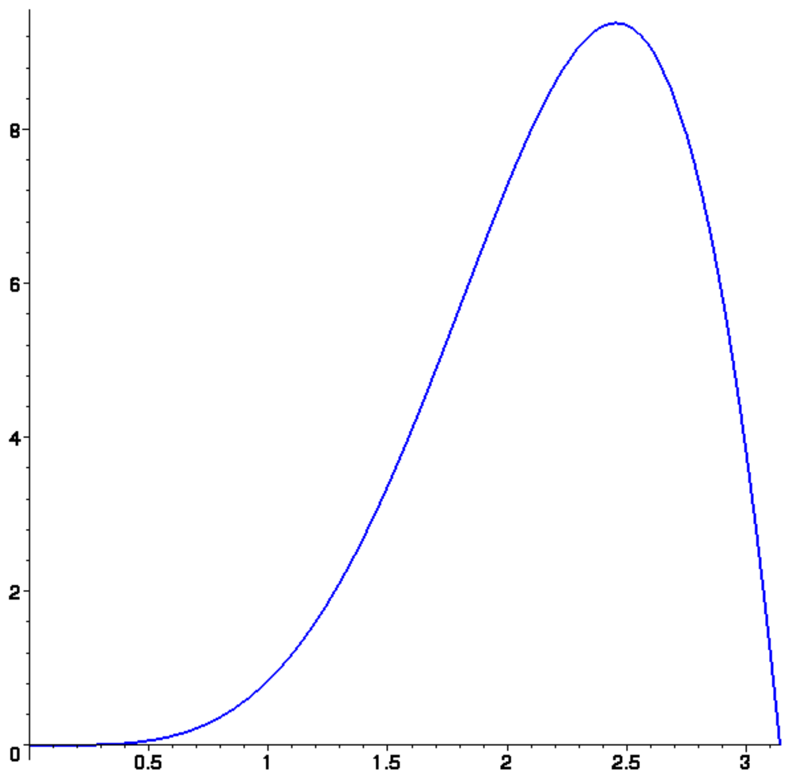
First 20 eigenvalues and 20 b's are given below. We can use them to check orthogonality. For first 5 values on n **and** m, **we have:**

$$\begin{bmatrix} 0.999999999999999, & -0.1509\ 10^{-14}, & -0.9962\ 10^{-15}, & 0.40753\ 10^{-14}, & -0.16247\ 10^{-14} \\ -0.1509\ 10^{-14}, & 0.999999999999999, & 0.35415\ 10^{-14}, & -0.77691\ 10^{-14}, & 0.29631\ 10^{-14} \\ -0.9962\ 10^{-15}, & 0.35415\ 10^{-14}, & 1.00000000000000, & 0.116848\ 10^{-13}, & -0.30389\ 10^{-14} \\ 0.40753\ 10^{-14}, & -0.77691\ 10^{-14}, & 0.116848\ 10^{-13}, & 1.00000000000000, & -0.37649\ 10^{-14} \\ -0.16247\ 10^{-14}, & 0.29631\ 10^{-14}, & -0.30389\ 10^{-14}, & -0.37649\ 10^{-14}, & 0.999999999999998 \end{bmatrix}$$

b[1] := 1.87936734845247	lambda[1] := 0.787637294164865
b[2] := 1.70257414190605	lambda[2] := 1.67160562540478
b[3] := 1.63453465475276	lambda[3] := 2.61621358543049
b[4] := 1.60686261795743	lambda[4] := 3.58655274644395
b[5] := 1.59365657285845	lambda[5] := 4.56859174556457
b[6] := 1.58648178184738	lambda[6] := 5.55667753696976
b[7] := 1.58219119230698	lambda[7] := 6.54823733019677
b[8] := 1.57943471512880	lambda[8] := 7.54196043868321
b[9] := 1.57756384259765	lambda[9] := 8.53711627308128
b[10] := 1.57623801837701	lambda[10] := 9.53326771828444
b[11] := 1.57526524869711	lambda[11] := 10.5301380799801
b[12] := 1.57453089542633	lambda[12] := 11.5275440291038
b[13] := 1.57396319618907	lambda[13] := 12.5253594436376
b[14] := 1.57351541884358	lambda[14] := 13.5234947861398
b[15] := 1.57315609589478	lambda[15] := 14.5218847749212
b[16] := 1.57286342204806	lambda[16] := 15.5204807127900
b[17] := 1.57262190736953	lambda[17] := 16.5192455471455
b[18] := 1.57242030660976	lambda[18] := 17.5181505928222
b[19] := 1.57225029824458	lambda[19] := 18.5171733006550
b[20] := 1.57210561874205	lambda[20] := 19.5162957027907

Example 1

$$f(x) = x^3 \sin(x)$$



A 20-term approximation:

$$u(x, t) = \sum_{i=1}^{20} a_i \sin(\lambda_i x) e^{\left(-\lambda_i^2 t\right)}$$

a[1] := 5.886131364786786

a[2] := -2.938976507601575

a[3] := -0.7131261828724985

a[4] := 1.094169264255642

a[5] := -0.7632510364002911

a[6] := 0.5655633427233659

a[7] := -0.4194929690238976

a[8] := 0.3254857221770207

a[9] := -0.2570186443987639

a[10] := 0.2087761417566782

a[11] := -0.1721186405473902

a[12] := 0.1446093842844635

a[13] := -0.1228946311423716

a[14] := 0.1058547113531052

a[15] := -0.09199048476748587

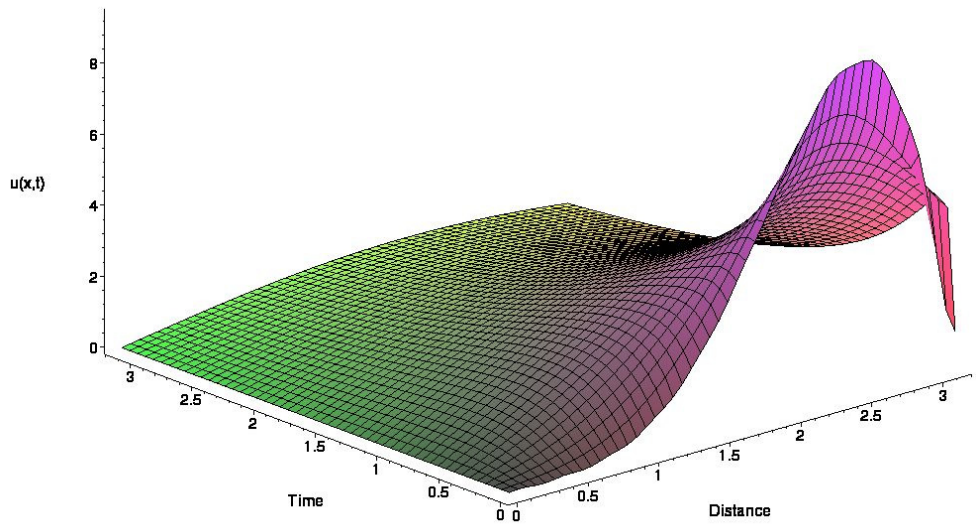
a[16] := 0.08074638757987318

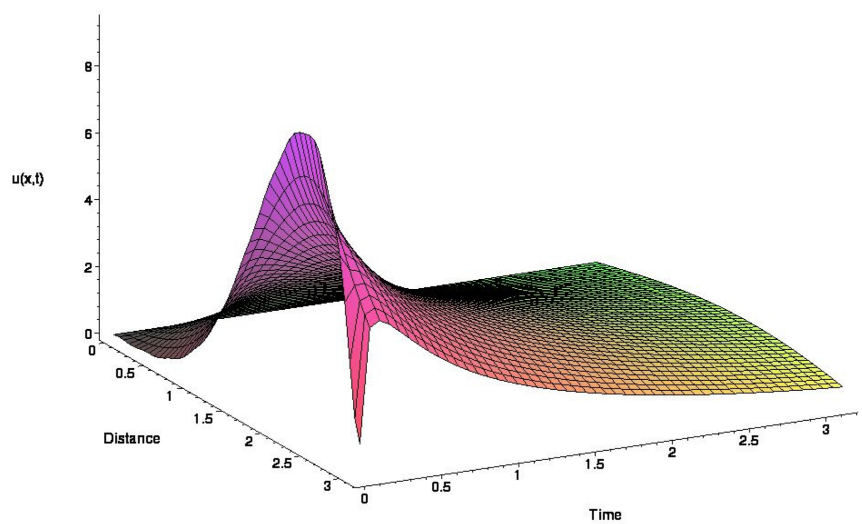
a[17] := -0.07137613871681960

a[18] := 0.06358261109020068

a[19] := -0.05696200470968955

a[20] := 0.05134488225526947


u(x,t)
Time
Distance


u(x,t)
Distance
Time