May 26, 2009

MAE 105 Homework #9
Due: Tuesday, 06/02/09

Problem 1

Consider the following nonhomogeneous diffusion equation:

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + x^2 t = 0\,, \qquad 0 < x < 1\,, \tag{1}$$

with boundary conditions

$$u(0,t) = 0\,, \qquad u(1,t) = 0\,, \tag{2}$$

and the initial condition

$$u(x,0) = x\,(x-1)\,. \tag{3}$$

(a) (1 Point) By considering the following eigenvalue problem:

$$\frac{d^2\phi}{dx^2} + \lambda^2\phi = 0\,,$$

and appropriate boundary conditions, find the eigenfunctions that can be used to solve (1) with boundary conditions (2).

(b) (3 Points) Set $u(x,t) \approx \sum_{n=0}^{\infty} A_n(t)\phi_n(x)$, $x\,(x-1) \approx \sum_{n=0}^{\infty} B_n\phi_n(x)$, and $x^2 \approx \sum_{n=0}^{\infty} C_n\phi_n(x)$, where $\phi_n(x)$ are the eigenfunctions that you obtained in (a), and reduce the PDE (1) into ODE's for the coefficients $A_n(t)$.

(c) (2 Points) Solve the ODE's to find $A_n(t)$, which must include an integration constant for each n.

(d) (2 Points) Use Fourier series of the initial condition that you worked out in (b) above to obtain the constant of integration for each n and write down the complete series solution of (1) subject to (2) and (3).

Note: You need the initial condition (3) to evaluate $A_n(0)$.

Problem 2

Consider now the following PDE:

$$\frac{\partial u}{\partial t} + x\,\frac{\partial u}{\partial x} = ux\,, \qquad t > 0. \tag{4}$$

(a) (1.5 Points) Find the characteristic curves, on which (4) becomes

$$\frac{du}{dt} = ux\,. \tag{5}$$

(b) (1.5 Points) If the initial condition is $u(x_0, 0) = \ln x_0$, find the final solution, $u = u(x, t)$ explicitly.

Problem 3

Consider the following nonhomogeneous wave equation:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + e^t\, x = 0, \qquad 0 < x < \pi, \tag{6}$$

with boundary conditions

$$u(0, t) = 0, \qquad u(\pi, t) = 0, \tag{7}$$

and the initial conditions

$$u(x, 0) = 0 \qquad \frac{\partial u}{\partial t}(x, 0) = 0. \tag{8}$$

(a) (1 Points) By considering the following eigenvalue problem:

$$\frac{d^2\phi}{dx^2} + \lambda^2 \phi = 0,$$

and appropriate boundary conditions, find the eigenfunctions that can be used to solve (6) with boundary conditions (7).

(b) (3 Points) Set $u(x, t) \approx \sum_{n=0}^{\infty} A_n(t)\phi_n(x)$ and $x \approx \sum_{n=0}^{\infty} B_n \phi_n(x)$, where $\phi_n(x)$ are the eigenfunctions that you obtained in (a), and reduce the PDE (6) into ODE's for the coefficients $A_n(t)$.

(c) (3 Points) Solve the ODE's to find $A_n(t)$, and obtain the complete series solution of (6) subject to (7) and (8).

Note: You need the initial condition (8) to evaluate the constants of integration of the 2nd order ODE that defines $A_n(t)$.