May 19, 2009

MAE 105 Homework #8
Due: Tuesday, 05/26/09

PROBLEM 1 (6 Points):

Consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 \tag{1}$$

in an infinite domain, $-\infty < x < \infty,\ t > 0$.

(a) (3 Points) Draw the x, t-plane and the characteristics which pass through $(x = 0, t = 1)$, $(x = 1, t = 1)$, $(x = 2, t = 1.5)$, and $(x = 2, t = 3)$. Sketch the following initial conditions right under the x, t-plane and explain why at the above points (or any other point in the x, t-plane), the corresponding $u(x, t)$ is completely defined by the initial conditions:

$$u(x, 0) = |\cos(\pi x)|, \quad 0.5 < x < 3.5,$$

$$= 0 \quad otherwise,$$

$$\frac{\partial u}{\partial t}(x, 0) = \begin{cases} x & \text{for } 0 < x < 1/2 \\ 1 - x & \text{for } 1/2 < x < 1 \\ |\sin(\pi x)| & \text{for } 1 < x < 4 \\ 0 & otherwise. \end{cases}$$

Note: The general solution of the wave equation for *infinite domain* is

$$u(x, t) = \tfrac{1}{2}[f(x - t) + f(x + t)] + \tfrac{1}{2}\int_{x-t}^{x+t} g(\bar{x})d\bar{x}. \tag{2}$$

(b) (3 Points) Using the relevant characteristics find the values of $u(x, t)$ at the four points stated in part (a) above.

PROBLEM 2 (5 Points):

Consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0$$

in a finite domain, $0 < x < 1,\ t > 0$.

(a) (1 Point) Draw the x, t-plane and the characteristics which pass through $(x = 0, t = 0)$ and $(x = 1, t = 0)$. Explain why for points inside the triangle formed by these characteristics and the x-axis, the value of $u(x, t)$ is completely defined by the following initial conditions independently of the boundary conditions:

$$u(x, 0) = \sin(\pi x), \qquad \frac{\partial u}{\partial t}(x, 0) = \begin{cases} x & \text{for } 0 < x < 1/2 \\ 1 - x & \text{for } 1/2 < x < 1. \end{cases}$$

(b) (1 Point) The characteristics of this problem are straight lines of the form $x = x_0 \pm t$. Find the equations of the two (right and left) characteristics that pass through the point $(x = 0, t = 1/4)$. Draw these characteristics and clearly identify each with its equation.

(c) (1 Point) Find the equations of the two (right and left) characteristics that pass through the point $(x = 1, t = 1/4)$. Draw these characteristics and clearly identify each with its equation.

(d) (1 Point) Let the boundary conditions be given by

$$u(0, t) = u(1, t) = 0 .$$

Extend the initial conditions such that these boundary conditions are satisfied when the general solution (2) of infinite domain is used to obtain $u(x, t)$ in the finite domain $0 < x < 1$ for all time $t > 0$; just sketch the extension for $-3 < x < 3$.

(e) (1 Point) Draw the characteristics through the point $(x = 1/4, t = 1.5)$ and find the value of $u(x, t)$ at this point.

[To receive full credit you must show all necessary details.]