May 12, 2009

MAE 105 Homework #7
Due: Tuesday, 05/19/09

PROBLEM 1 (2 Points):

On the same graph, plot the following three functions:

$$y_1 = \cos(x), \qquad 0 < x < \pi/2, \tag{*}$$

$$y_2 = \cos(x - \pi/2), \qquad \pi/2 < x < \pi, \tag{**}$$

$$y_3 = \cos(x + \pi/2), \qquad -\pi/2 < x < 0, \tag{***}$$

Note that y_2 and y_3 are obtained from y_1 by **rigid translations.** What is the respective magnitude of these translations?

PROBLEM 2 (2 Points):

Consider the following first order differential equation:

$$\frac{dx}{dt} = 3xt^2. \tag{1}$$

In the x, t –plane, the solution of (1) is a curve $x = x(t)$.

(a) (1 Point) Write (1) as

$$\frac{dx}{x} = 3t^2\, dt \tag{2}$$

and integrate to obtain a general solution which includes a constant of integration.

(b) (1 Point) If at $t = 0$, $x(0) = x_0$, show that the solution can be expressed as

$$x = x_0 e^{t^3}. \tag{3}$$

PROBLEM 3 (2 Points):

Consider now the following PDE:

$$\frac{\partial u}{\partial t} + 3xt^2 \frac{\partial u}{\partial x} = 0, \quad t > 0. \tag{4}$$

(a) (1.5 Points) Use the result in (1), (2), and (3) to reduce (4) into

$$\frac{du}{dt} = 0, \quad \textit{on curves} \quad x = x_0 e^{t^3}, \tag{5}$$

and integrate to get the solution $u(x, t) = u_0 = \textit{constant}$, when x and t are related by (3).

(b) (1.5 Points) If the initial condition is $u(x_0, 0) = \ln x_0$, show that the final solution is given by [show all steps]

$$u(x, t) = \ln x - t^3.$$

PROBLEM 4 (5 Points):

Consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0$$

in a finite domain, $0 < x < 1$, $t > 0$.

(a) (1 Point) Draw the x, t-plane and the characteristics which pass through $(x = 0, t = 0)$ and $(x = 1, t = 0)$. Explain why for points inside the triangle formed by these characteristics and the x-axis, the value of $u(x, t)$ is completely defined by the following initial conditions independently of the boundary conditions:

$$u(x, 0) = \sin(\pi x), \qquad \frac{\partial u}{\partial t}(x, 0) = \begin{cases} x & \text{for } 0 < x < 1/2 \\ 1 - x & \text{for } 1/2 < x < 1 . \end{cases}$$

Note: The general solution of the wave equation for *infinite domain* is

$$u(x, t) = \tfrac{1}{2}[f(x - t) + f(x + t)] + \tfrac{1}{2}\int_{x-t}^{x+t} g(\bar{x}) d\bar{x} . \qquad (6)$$

(b) (1 Point) The characteristics of this problem are straight lines of the form $x = x_0 \pm t$. Find the equations of the two (right and left) characteristics that pass through the point $(x = 0, t = 1/4)$. Draw these characteristics and clearly identify each with its equation.

(c) (1 Point) Find the equations of the two (right and left) characteristics that pass through the point $(x = 1, t = 1/4)$. Draw these characteristics and clearly identify each with its equation.

(d) (1 Point) Let the boundary conditions be given by

$$u(0, t) = u(1, t) = 0 .$$

Extend the initial conditions such that these boundary conditions are satisfied when the general solution (6) of infinite domain is used to obtain $u(x, t)$ in the finite domain $0 < x < 1$ for all time $t > 0$; just sketch the extension for $-3 < x < 3$.

(e) (1 Point) Draw the characteristics through the point $(x = 1/4, t = 1.5)$ and find the value of $u(x, t)$ at this point.

[To receive full credit you must show all necessary details.]