May 05, 2009

MAE 105 Homework #6
Due: Tuesday, 05/12/09

PROBLEM 1 (5 Points):

Consider the diffusion equation in a rectangular region,

$$\frac{\partial u(x,\, y,\, t)}{\partial t} - \nabla^2 u(x,\, y,\, t) = 0\,, \qquad 0 < x < L\,, \qquad 0 < y < H\,, \qquad t > 0\,, \tag{1}$$

with boundary conditions

$$u(x,\, 0,\, t) = 0\,, \qquad u(x,\, H,\, t) + \alpha\, \frac{\partial u(x,\, H,\, t)}{\partial y} = 0\,, \qquad u(0,\, y,\, t) = 0\,, \qquad \frac{\partial u}{\partial x}(L,\, y,\, t) = 0\,,$$

where α is a constant, and the initial condition,

$$u(x,\, y,\, 0) = \beta(x,\, y)\,.$$

(a) (0.5 Point) Set $u(x,\, y,\, t) = h(t)\,\phi(x,\, y)$. Find the ODE for $h(t)$ and the PDE for $\phi(x,\, y)$.

(b) (0.5 Point) Find the general solution for $h(t)$ which decays in time.

(c) (1 Point) Set $\phi(x,\, y) = f(x)\, g(y)$. Find the ODE's for $f(x)$ and $g(y)$, and the boundary conditions for each of these functions.

(d) (0.5 Point) For part (d) only, set $\alpha = -1$. Sketch how to find the eigenvalues for $g(y)$ graphically.

For the remaining part of this PROBLEM (parts (e) to (i)), set $\alpha = 0$.

(e) (0.5 Point) Apply the B.C.'s and find the general solution fo $f(x)$.

(f) (0.5 Point) Apply the B.C.'s and find the general solution fo $g(y)$.

(g) (0.5 Point) Find the eigenvalues and eigenfunctions associated with $\phi(x,\, y)$.

(h) (0.5 Point) Write out the infinite series solution for $u(x,\, y,\, t)$.

(i) (0.5 Point) Use the following expression for the initial condition to find the constants in the infinite series solution:

$$\beta(x,\, y) = 3\sin(3\,\pi\, y/H)\sin(2.5\,\pi\, x/L)\,.$$

PROBLEM 2 (3 Points):

Consider the following ODE:

$$y' + \alpha\, y = 0\,, \qquad x > 0\,, \tag{*}$$

where prime denotes differentiation with respect x, and α is a given constant. Consider a series solution of the form

$$y = \sum_{n=0}^{\infty} a_n\, x^n\,.$$

(a) Substitute the infinite series into the ODE and find all the a_n's. Write down the final series solution.

(b) Directly integrate the original ODE (*) and find a closed-form solution.

(c) Are the two solutions the same?

PROBLEM 3 (3 Points):

Consider the following ODE:

$$x\,y' + \alpha\,y = 0\,, \qquad x > 0\,, \tag{**}$$

where prime denotes differentiation with respect x, and α and A_0 are given constants. Consider a series solution of the form

$$y = \sum_{n=0}^{\infty} a_n\,x^n\,.$$

(a) Substitute the infinite series into the ODE and try to find all the a_n's, in the same manner as in Problem 1. What do you get, now?

(b) Directly integrate the original ODE (**) and find a closed-form solution.

(c) Consider now the following series solution:

$$y = x^r \sum_{n=0}^{\infty} a_n\,x^n\,.$$

Substitute the infinite series into the ODE (**) and show that the solution is now given by $y = A_0\,x^{-\alpha}$, where A_0 is a constant. [You must show what happens to all a_n's!]

PROBLEM 4 (3 Points):

Consider the following ODE:

$$x^2\,y' + \alpha\,y = 0\,, \qquad x > 0\,, \tag{***}$$

where prime denotes differentiation with respect x, and α and A_0 are given constants. Consider a series solution of the form

$$y = \sum_{n=0}^{\infty} a_n\,x^{-n}\,.$$

(a) Substitute the infinite series into the ODE and find all the a_n's. Write down the final series solution.

(b) Directly integrate the original ODE (***) and find a closed-form solution.

(c) Are the two solutions the same?

Note 1: To receive full credit, *all steps must be neatly shown.* Writing down the final results will receive no credit.

Note 2: Homeworks must be turned in at the start of due-date class. Late homeworks will be graded but *will receive zero credit.*