P1. Consider the following modification of the LQR problem discussed in class, where we seek to determine a state feedback controller \( u(t) = Kx(t) \) that stabilizes the system

\[
\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0.
\]

while minimizing the following cost function

\[
J := \int_0^\infty e^{2\alpha t} [x(t)^TQx(t) + u(t)^TRu(t)] \, dt
\]

where \( \alpha > 0 \) is a given constant. As for the standard LQR problem, assume that \( Q \succeq 0 \) and \( R \succ 0 \) and that \( (A, B) \) stabilizable, and answer the following questions:

a) [1 mark] What is the impact of the exponential term in the integrand? How does it compare with the standard LQR cost? (Hint: Provide a qualitative discussion)

b) [2 marks] Show that the auxiliary state and control vectors

\[
\tilde{x}(t) = e^{\alpha t} x(t), \quad \tilde{u}(t) = e^{\alpha t} u(t)
\]

satisfy the differential equation

\[
\dot{\tilde{x}}(t) = (A + \alpha I)\tilde{x}(t) + B\tilde{u}(t), \quad \tilde{x}(0) = x_0
\]

and that

\[
J := \int_0^\infty \tilde{x}(t)^TQ\tilde{x}(t) + \tilde{u}(t)^TR\tilde{u} \, dt
\]

c) [4 marks] Using item b) and what you know about the LQR problem show how to compute the optimal state feedback controller \( \tilde{u}(t) = \tilde{K}\tilde{x}(t) \). You do not have to prove anything, but you should list all steps and the assumptions you used in each step.
d) [2 marks] Show how you can use the optimal controller in item c) to produce an optimal controller \( u(t) = Kx(t) \) that solves the original problem.

e) [1 mark] What is the impact of \( \alpha > 0 \) on the original assumptions? Did you need to modify them in any way?

P2. When analyzing the LQR problem we have assumed at the onset that a stabilizing solution was sought. However, the stabilizing solution will not always be an optimal solution. This problem will help you understand that. In the following items assume that \( x(0) = x_0 \).

a) [2 marks] Without stability as a requirement, what is the optimal solution to the following control problem

\[
\min_K \left\{ \int_0^\infty u(t)^T u(t) \, dt : \quad \dot{x}(t) = Ax(t) + Bu(t), \quad u(t) = Kx(t) \right\}
\]

(Hint: you do not need any fancy calculations to solve this!)

b) [2 marks] When \( A \) is Hurwitz, then the optimal solution is also stabilizing. To see that compute the stabilizing solution to the Riccati equation and the associated gain \( K \) for \( A = -1, B = 1 \).

c) [4 marks] When \( A \) is not Hurwitz, then the optimal solution is not stabilizing. To see that compute the stabilizing solution to the Riccati equation and the associated gain \( K \) for \( A = 1, B = 1 \). Compute also the “non-stabilizing” solution to the Riccati equation and the associated gain \( K \). Which one is optimal?

d) [4 marks] In order to make peace with our stability assumption compute the optimal solution to the LQR problem

\[
\min_K \left\{ \int_0^\infty \epsilon x(t)^T x(t) + u(t)^T u(t) \, dt : \quad \dot{x}(t) = Ax(t) + Bu(t), \quad u(t) = Kx(t) \right\}
\]

when \( A = 1, B = 1 \) and \( \epsilon > 0 \). Do that symbolically as a function of \( \epsilon \). Is this solution stabilizing? Compute the limit of this solution as \( \epsilon \to 0 \). Compare the answer with the solution to part c).

P3. Consider the LTI system associated with the transfer function

\[
H(s) = \frac{s - 1}{s - 3}.
\]

a) [5 marks] Using any “classical” control technique (root-locus, bode diagram, Nyquist criterion, etc) find a controller that can stabilize \( H(s) \). What is the order of the controller you found? Is the controller itself stable?

(Hint #1: Do not cancel zeros on the right hand side of the complex plane!)

(Hint #2: I can do it with a single pole!)

b) [2 marks] Compute a state space representation \((A, B_u, C_y, D_u)\) for \( H(s) \).
c) [2 bonus marks] Show that if you compute a controller $\tilde{C}(s)$ that stabilizes the strictly proper system

$$\dot{x} = Ax + B_u u,$$
$$\tilde{y} = C_y x$$

then the controller $C(s) = [I + \tilde{C}(s)D_u]^{-1}\tilde{C}(s)$ stabilizes the original system with transfer function $H(s)$.

d) [5 marks] With the above item in mind, let the input $u(t)$ of the system be perturbed by $w(t)$ and the measurement output $\tilde{y}(t)$ be perturbed by $v(t)$ in the form

$$\dot{x} = Ax + B_u(u + w)$$
$$\tilde{y} = C_y x + v.$$

Assuming that $w(t)$ and $v(t)$ are independent Gaussian zero-mean white-noise with unitary variance, find observer-based optimal controllers that minimize the cost function

$$J = \lim_{t \to \infty} E[z(t)^T z(t)]$$

where

$$z = \begin{pmatrix} \tilde{y} - v \\ \rho u \end{pmatrix}$$

for choices of $\rho = \{10^{-6}, 10^{-3}, 1, 10^{3}, 10^{6}\}$.

e) [5 marks] What is the impact of the different choices of $\rho$ on the cost function, the closed loop eigenvalues, the controller poles, the controller zeros? Is any of these controllers stable? Do you think you can find a stable stabilizing controller? Why?

IMPORTANT: Answer the items related to the controller with respect to $C(s)$, that is the controller of the original system with transfer function $H(s)$, not $\tilde{C}(s)$!

f) [5 marks] Solve the problem again for

$$H(s) = \frac{s + 1}{s - 3}$$

What makes this problem easier or harder than the previous one? Could you find a stable controller using classical techniques? How does it compare with the LQG controllers?