MAE 280B – Linear Control Design – Winter 2010

Midterm

Instructions:

- Due on 02/12/2010 on my office (EBU I 1602) by 5:00 PM;
- Use Matlab;
- You get marks for clarity;
- You loose marks for obscurantism;
- Good luck!

P1. Consider the following modification of the LQR problem discussed in class, where we seek to determine a state feedback controller $u(t) = Kx(t)$ that stabilizes the system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0.$$

while minimizing the following cost function

$$J := \int_0^\infty e^{2\alpha t}[x(t)^T Q x(t) + u(t)^T R u(t)]\, dt$$

where $\alpha > 0$ is a given constant. As for ths standard LQR problem, assume that $Q \succeq 0$ and $R \succ 0$ and that (A, B) stabilizable, and answer the following questions:

a) [1 mark] What is the impact of the exponential term in the integrand? How does it compare with the standard LQR cost?
(Hint: Provide a qualitative discussion)

b) [2 marks] Show that the auxiliary state and control vectors

$$\tilde{x}(t) = e^{\alpha t}x(t), \qquad \tilde{u}(t) = e^{\alpha t}u(t)$$

satisfy the differential equation

$$\dot{\tilde{x}}(t) = (A + \alpha I)\tilde{x}(t) + B\tilde{u}(t), \quad \tilde{x}(0) = x_0$$

and that

$$J := \int_0^\infty \tilde{x}(t)^T Q \tilde{x}(t) + \tilde{u}(t)^T R \tilde{u}\, dt$$

c) [4 marks] Using item b) and what you know about the LQR problem show how to compute the optimal state feedback controller $\tilde{u}(t) = \tilde{K}\tilde{x}(t)$. You do not have to prove anything, but you should list all steps and the assumptions you used in each step.

d) [2 marks] Show how you can use the optimal controller in item c) to produce an optimal controller $u(t) = Kx(t)$ that solves the original problem.

e) [1 mark] What is the impact of $\alpha > 0$ on the original assumptions? Did you need to modify them in any way?

P2. When analyzing the LQR problem we have assumed at the onset that a *stabilizing solution* was sought. However, the stabilizing solution will not always be an *optimal* solution. This problem will help you understand that. In the following items assume that $x(0) = x_0$.

a) [2 marks] Without stability as a requirement, what is the optimal solution to the following control problem

$$\min_K \left\{ \int_0^\infty u(t)^T u(t)\, dt : \quad \dot{x}(t) = Ax(t) + Bu(t), \quad u(t) = Kx(t) \right\}$$

(Hint: you do not need any fancy calculations to solve this!)

b) [2 marks] When A is Hurwitz, then the optimal solution is also stabilizing. To see that compute the stabilizing solution to the Riccati equation and the associated gain K for $A = -1$, $B = 1$.

c) [4 marks] When A is not Hurwitz, then the optimal solution is not stabilizing. To see that compute the stabilizing solution to the Riccati equation and the associated gain K for $A = 1$, $B = 1$. Compute also the "non-stabilizing" solution to the Riccati equation and the associated gain K. Which one is optimal?

d) [4 marks] In order to make peace with our stability assumption compute the optimal solution to the LQR problem

$$\min_K \left\{ \int_0^\infty \epsilon\, x(t)^T x(t) + u(t)^T u(t)\, dt : \quad \dot{x}(t) = Ax(t) + Bu(t), \quad u(t) = Kx(t) \right\}$$

when $A = 1$, $B = 1$ and $\epsilon > 0$. Do that symbolically as a function of ϵ. Is this solution stabilizing? Compute the limit of this solution as $\epsilon \to 0$. Compare the answer with the solution to part c).

P3. Consider the LTI system associated with the transfer function

$$H(s) = \frac{s-1}{s-3}.$$

a) [5 marks] Using any "classical" control technique (root-locus, bode diagram, Nyquist criterion, etc) find a controller that can stabilize $H(s)$. What is the order of the controller you found? Is the controller itself stable?
(Hint #1: Do not cancel zeros on the right hand side of the complex plane!)
(Hint #2: I can do it with a single pole!)

b) [2 marks] Compute a state space representation (A, B_u, C_y, D_u) for $H(s)$.

c) [2 bonus marks] Show that if you compute a controller $\tilde{C}(s)$ that stabilizes the strictly proper system

$$\begin{aligned}\dot{x} &= Ax + B_u u,\\ \tilde{y} &= C_y x\end{aligned}$$

then the controller $C(s) = [I + \tilde{C}(s)D_u]^{-1}\tilde{C}(s)$ stabilizes the original system with transfer function $H(s)$.

d) [5 marks] With the above item in mind, let the input $u(t)$ of the system be perturbed by $w(t)$ and the measurement output $\tilde{y}(t)$ be perturbed by $v(t)$ in the form

$$\begin{aligned}\dot{x} &= Ax + B_u(u + w)\\ \tilde{y} &= C_y x + v.\end{aligned}$$

Assuming that $w(t)$ and $v(t)$ are independent Gaussian zero-mean white-noise with unitary variance, find observer-based optimal controllers that minimize the cost function

$$J = \lim_{t\to\infty} E\left[z(t)^T z(t)\right]$$

where

$$z = \begin{pmatrix} \tilde{y} - v \\ \rho u \end{pmatrix}$$

for choices of $\rho = \{10^{-6}, 10^{-3}, 1, 10^3, 10^6\}$.

e) [5 marks] What is the impact of the different choices of ρ on the cost function, the closed loop eigenvalues, the controller poles, the controller zeros? Is any of these controllers stable? Do you think you can find a stable stabilizing controller? Why? IMPORTANT: Answer the items related to the controller with respect to $C(s)$, that is the controller of the original system with transfer function $H(s)$, not $\tilde{C}(s)$!

f) [5 marks] Solve the problem again for

$$H(s) = \frac{s+1}{s-3}$$

What makes this problem easier or harder than the previous one? Could you find a stable controller using classical techniques? How does it compare with the LQG controllers?