9 The LQR Problem Revisited

Problem: Compute a state feedback controller

$$u(t) = Kx(t)$$

that stabilizes the closed loop system and minimizes

$$J := \lim_{t \to \infty} E\left[z(t)^T z(t)\right]$$

for the LTI system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_u u(t) + B_w w(t), \quad x(0) = 0 \\ z(t) &= C_z x(t) + D_{zu} u(t). \end{aligned}$$

Assumptions:

a) $D_{zu}^T D_{zu} \succ 0$;

b) (A, B_u) stabilizable;

c) $w(t)$ is a Gaussian white noise vector with zero mean and covariance $W \succ 0$.

Solution: As before, restate the cost function with the help of the Gramian

$$J = \mathrm{trace}\left(P B_w W B_w^T\right)$$

where

$$(A + B_u K)^T P + P(A + B_u K) + (C_z + D_{zu})^T (C_z + D_{zu}) = 0.$$

Use the Comparison Lemma to conclude that for any K such that $(A + B_u K)$ is Hurwitz then

$$Y \succeq P \qquad \Longrightarrow \qquad \mathrm{trace}\left(Y B_w W B_w^T\right) \geq J$$

where Y satisfies the inequality

$$(A + B_u K)^T Y + Y(A + B_u K) + (C_z + D_{zu} K)^T (C_z + D_{zu} K) \preceq 0.$$

9.1 LQR revisited: solution by congruence + change-of-variables

Recipe I (congruence + change-of-variables):
First apply Schur complement

$$(A + B_u K)^T Y + Y(A + B_u K) + (C_z + D_{zu} K)^T (C_z + D_{zu} K) \preceq 0$$
$$\Updownarrow$$
$$\begin{bmatrix} (A + B_u K)^T Y + Y(A + B_u K) & C_z^T + K^T D_{zu}^T \\ C_z + D_{zu} K & -I \end{bmatrix} \preceq 0$$

Under the assumption that (A, B_u) is stabilizable and $D_{zu}^T D_{zu} \succ 0$ we have that $Y \succeq P \succ 0$ so that we can apply the congruence transformation

$$\begin{bmatrix} Y^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} (A + B_u K)^T Y + Y(A + B_u K) & C_z^T + K^T D_{zu}^T \\ C_z + D_{zu} K & -I \end{bmatrix} \begin{bmatrix} Y^{-1} & 0 \\ 0 & I \end{bmatrix} \preceq 0$$
$$\Updownarrow$$
$$\begin{bmatrix} (A + B_u K) Y^{-1} + Y^{-1} (A + B_u K)^T & Y^{-1} C_z^T + Y^{-1} K^T D_{zu}^T \\ C_z Y^{-1} + D_{zu} K Y^{-1} & -I \end{bmatrix} \preceq 0$$

Apply change-of-variables

$$X := Y^{-1}, \qquad L := K Y^{-1}$$

to obtain the LMI

$$\begin{bmatrix} AX + XA^T + B_u L + L^T B_u^T & X C_z^T + L^T D_{zu}^T \\ C_z X + D_{zu} L & -I \end{bmatrix} \preceq 0$$

As for the cost function

$$\operatorname{trace}\left(X^{-1} B_w W B_w^T\right) = \operatorname{trace}\left(Y B_w W B_w^T\right) \geq J$$

Now introduce an extra variable Z such that

$$Z \succeq B_w^T X^{-1} B_w$$

Therefore

$$\operatorname{trace}(ZW) = \operatorname{trace}\left(B_w^T X^{-1} B_w W\right) \geq J$$

Using Schur Complement

$$Z \succeq B_w^T X^{-1} B_w \qquad \Leftrightarrow \qquad \begin{bmatrix} Z & B_w^T \\ B_w & X \end{bmatrix} \succeq 0.$$

9.1.1 Summary: LQR revisited (first form)

The state feedback controller

$$u(t) = Kx(t), \qquad K = LX^{-1}$$

where $X \in \mathbb{S}^n$, $L \in \mathbb{R}^{m \times n}$ and $Z \in \mathbb{S}^r$ solve the SDP

$$\begin{aligned} \min_{X \in \mathbb{S}^n, L \in \mathbb{R}^{m \times n}, Z \in \mathbb{S}^r} \quad & \operatorname{trace}(ZW) \\ \text{s.t.} \quad & \begin{bmatrix} AX + XA^T + B_u L + L^T B_u^T & XC_z^T + L^T D_{zu}^T \\ C_z X + D_{zu} L & -I \end{bmatrix} \preceq 0, \\ & \begin{bmatrix} Z & B_w^T \\ B_w & X \end{bmatrix} \succeq 0, \quad X \succ 0. \end{aligned}$$

stabilizes the closed loop system and minimizes the cost function

$$J := \lim_{t \to \infty} E\left[z(t)^T z(t)\right]$$

for the LTI system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_u u(t) + B_w w(t), \quad x(0) = 0 \\ z(t) &= C_z x(t) + D_{zu} u(t) \end{aligned}$$

under the assumptions

a) $D_{zu}^T D_{zu} \succ 0$;

b) (A, B_u) stabilizable;

c) $w(t)$ is a Gaussian white noise vector with zero mean and covariance $W \succ 0$.

9.2 Elimination Lemma

Let $x \in \mathbb{R}^n$, $\mathcal{Q} \in \mathbb{S}^n$, $\mathcal{B} \in \mathbb{R}^{m \times n}$ and $\mathcal{B} \in \mathbb{R}^{p \times n}$ such that $\operatorname{rank}(\mathcal{B}) < n$ and $\operatorname{rank}(\mathcal{C}) < n$. The following statements are equivalent:

i) ${\mathcal{B}^{\perp}}^T \mathcal{Q} \mathcal{B}^{\perp} \prec 0$ and ${\mathcal{C}^{\perp}}^T \mathcal{Q} \mathcal{C}^{\perp} \prec 0$

ii) $\exists\, \mu \in \mathbb{R} : \mathcal{Q} - \mu \mathcal{B}^T \mathcal{B} \prec 0$, and $\mathcal{Q} - \mu \mathcal{C}^T \mathcal{C} \prec 0$.

iii) $\exists\, \mathcal{X} \in \mathbb{R}^{p \times m} : \mathcal{Q} + \mathcal{C}^T \mathcal{X} \mathcal{B} + \mathcal{B}^T \mathcal{X}^T \mathcal{C} \prec 0$.

Remark: Proofs can be found in Skelton, Iwasaki and Grigoriadis (Theorem 2.3.12, p. 29.) or Boyd, El Gahoui, Feron & Balakrishnan (p. 32). None as nice as the next one :).

Proof:
$ii)$ and $iii) \Rightarrow i)$ is as in Finsler's Lemma, just multiply by $\mathcal{B}^{\perp}$ and $\mathcal{C}^{\perp}$.

$i) \Rightarrow ii)$ use Finsler's Lemma to show that $i)$ implies the existence of scalars μ_1 and μ_2 such that

$$\mathcal{Q} - \mu_1 \mathcal{B}^T \mathcal{B} \prec 0, \qquad \mathcal{Q} - \mu_2 \mathcal{C}^T \mathcal{C} \prec 0.$$

Then $\mu = \max\{\mu_1, \mu_2\}$ is such that $ii)$ holds for some μ.

$ii) \Rightarrow iii)$ requires an explicit construction of $\mathcal{X}$.

9.2.1 Construction of $\mathcal{X}$

Assume that $ii)$ holds. If $\mu \leq 0$ then $\mathcal{X} = 0$ solves the problem. So assume $\mu > 0$. Now let $\rho^2 = \mu$ so that

$$\mathcal{Q} - \rho^2 \mathcal{B}^T \mathcal{B} \prec 0, \qquad \qquad \mathcal{Q} - \rho^2 \mathcal{C}^T \mathcal{C} \prec 0.$$

Define

$$\Phi := \mathcal{Q} - \rho^2 \mathcal{B}^T \mathcal{B} - \rho^2 \mathcal{C}^T \mathcal{C}$$

Then

$$\Phi + \rho^2 \mathcal{C}^T \mathcal{C} \prec 0, \qquad \qquad \Phi + \rho^2 \mathcal{B}^T \mathcal{B} \prec 0.$$

Also notice that $\Phi \prec 0$ because

$$\Phi := \frac{1}{2} \left[\left(\mathcal{Q} - 2\rho^2 \mathcal{B}^T \mathcal{B} \right) + \left(\mathcal{Q} - 2\rho^2 \mathcal{C}^T \mathcal{C} \right) \right]$$

and $2\rho^2 > \rho^2 = \mu$. Therefore, using Schur Complement

$$\Phi + \rho^2 \mathcal{C}^T \mathcal{C} \prec 0 \qquad \iff \qquad \begin{bmatrix} \Phi & \rho \mathcal{C}^T \\ \rho \mathcal{C} & -I \end{bmatrix} \prec 0.$$

Schur Complement one more time

$$\begin{bmatrix} \Phi & \rho \mathcal{C}^T \\ \rho \mathcal{C} & -I \end{bmatrix} \prec 0 \qquad \iff \qquad -I - \rho^2 \mathcal{C} \Phi^{-1} \mathcal{C}^T \prec 0.$$

Likewise

$$\Phi + \rho^2 \mathcal{B}^T \mathcal{B} \prec 0 \qquad \iff \qquad -I - \rho^2 \mathcal{B} \Phi^{-1} \mathcal{B}^T \prec 0.$$

We will now prove that

$$\mathcal{X} = \rho^2 \mathcal{Y} \qquad \qquad \mathcal{Y} = \rho^2 \mathcal{C} \, \Phi^{-1} \, \mathcal{B}^T$$

satisfies $iii)$.

With that in mind notice that

$$\begin{bmatrix} -I - \rho^2 \mathcal{B} \, \Phi^{-1} \, \mathcal{B}^T & \mathcal{Y}^T - \rho^2 \mathcal{B} \, \Phi^{-1} \, \mathcal{C}^T \\ \mathcal{Y} - \rho^2 \mathcal{C} \, \Phi^{-1} \, \mathcal{B}^T & -I - \rho^2 \mathcal{C} \, \Phi^{-1} \, \mathcal{C}^T \end{bmatrix} \prec 0.$$

Using Schur Complement this is equivalent to

$$\begin{bmatrix} \Phi & \rho\mathcal{B}^T & \rho\mathcal{C}^T \\ \rho\mathcal{B} & -I & \mathcal{Y}^T \\ \rho\mathcal{C} & \mathcal{Y} & -I \end{bmatrix} \prec 0$$

Using Schur Complement one more time

$$\begin{bmatrix} \Phi + \rho^2\mathcal{C}^T\mathcal{C} & \rho\mathcal{B}^T + \rho\mathcal{C}^T\mathcal{Y} \\ \rho\mathcal{B} + \rho\mathcal{Y}^T\mathcal{C} & -I + \mathcal{Y}^T\mathcal{Y} \end{bmatrix} = \begin{bmatrix} \mathcal{Q} - \rho^2\mathcal{B}^T\mathcal{B} & \rho\mathcal{B}^T + \rho\mathcal{C}^T\mathcal{Y} \\ \rho\mathcal{B} + \rho\mathcal{Y}^T\mathcal{C} & -I + \mathcal{Y}^T\mathcal{Y} \end{bmatrix} \prec 0$$

and once again

$$\mathcal{Q} - \rho^2\mathcal{B}^T\mathcal{B} + \rho^2(\mathcal{B}^T + \mathcal{C}^T\mathcal{Y})(I - \mathcal{Y}^T\mathcal{Y})^{-1}(\mathcal{B} + \mathcal{Y}^T\mathcal{C}) \prec 0$$

and $I - \mathcal{Y}^T\mathcal{Y} \succ 0$. Using the Lemma of the Matrix Inverse[2]

$$(I - \mathcal{Y}^T\mathcal{Y})^{-1} = I + \mathcal{Y}^T(I - \mathcal{Y}\mathcal{Y}^T)^{-1}\mathcal{Y}$$

so that

$$\begin{aligned} 0 \succ\ & \mathcal{Q} - \rho^2\mathcal{B}^T\mathcal{B} + \rho^2(\mathcal{B}^T + \mathcal{C}^T\mathcal{Y})(I - \mathcal{Y}^T\mathcal{Y})^{-1}(\mathcal{B} + \mathcal{Y}^T\mathcal{C}) \\ =\ & \mathcal{Q} + \rho^2\mathcal{C}^T\mathcal{Y}\mathcal{B} + \rho^2\mathcal{B}^T\mathcal{Y}^T\mathcal{C} + \rho^2\mathcal{C}^T\mathcal{Y}\mathcal{Y}^T\mathcal{C} \\ & \qquad + \rho^2(\mathcal{B} + \mathcal{C}^T\mathcal{Y})\mathcal{Y}^T(I - \mathcal{Y}^T\mathcal{Y})^{-1}\mathcal{Y}(\mathcal{B}^T + \mathcal{Y}^T\mathcal{C}) \end{aligned}$$

hence $iii)$ holds because

$$\rho^2\mathcal{C}^T\mathcal{Y}\mathcal{Y}^T\mathcal{C} + \rho^2(\mathcal{B} + \mathcal{C}^T\mathcal{Y})\mathcal{Y}^T(I - \mathcal{Y}^T\mathcal{Y})^{-1}\mathcal{Y}(\mathcal{B}^T + \mathcal{Y}^T\mathcal{C}) \succ 0.$$

[2] Verify that

$$\begin{aligned} \left[I + \mathcal{Y}^T(I - \mathcal{Y}\mathcal{Y}^T)^{-1}\mathcal{Y}\right](I - \mathcal{Y}^T\mathcal{Y}) &= I - \mathcal{Y}^T\mathcal{Y} + \mathcal{Y}^T(I - \mathcal{Y}\mathcal{Y}^T)^{-1}\mathcal{Y}(I - \mathcal{Y}^T\mathcal{Y}) \\ &= I - \mathcal{Y}^T\mathcal{Y} + \mathcal{Y}^T(I - \mathcal{Y}\mathcal{Y}^T)^{-1}(I - \mathcal{Y}\mathcal{Y}^T)\mathcal{Y} \\ &= I - \mathcal{Y}^T\mathcal{Y} + \mathcal{Y}^T\mathcal{Y} \\ &= I. \end{aligned}$$

9.3 LQR revisited: solution by Elimination Lemma

Recipe II (existence conditions):
As before we start with

$$Y \succ 0, \qquad \begin{bmatrix} (A + B_u K)^T Y + Y(A + B_u K) & C_z^T + K^T D_{zu}^T \\ C_z + D_{zu} K & -I \end{bmatrix} \prec 0$$

Note that we are using strict inequalities! Now rewrite

$$\underbrace{\begin{bmatrix} A^T Y + Y A & C_z^T \\ C_z & -I \end{bmatrix}}_{\mathcal{Q}} + \underbrace{\begin{bmatrix} Y B_u \\ D_{zu} \end{bmatrix}}_{\mathcal{B}^T} \underbrace{K}_{\mathcal{X}^T} \underbrace{\begin{bmatrix} I & 0 \end{bmatrix}}_{\mathcal{C}} + \underbrace{\begin{bmatrix} I \\ 0 \end{bmatrix}}_{\mathcal{C}^T} \underbrace{K^T}_{\mathcal{X}} \underbrace{\begin{bmatrix} B_u^T Y & D_{zu}^T \end{bmatrix}}_{\mathcal{B}} \prec 0.$$

We need a more general version of Finsler's Lemma.

9.3.1 Computing $\mathcal{B}^\perp$ and $\mathcal{C}^\perp$ under no assumptions

First compute

$$\mathcal{C}^\perp = \begin{bmatrix} I & 0 \end{bmatrix}^\perp = \begin{bmatrix} 0 \\ I \end{bmatrix}.$$

The problem of computing a basis for

$$0 = \mathcal{B}x = \begin{bmatrix} B_u^T Y & D_{zu}^T \end{bmatrix} x = \begin{bmatrix} B_u^T & D_{zu}^T \end{bmatrix} \begin{bmatrix} Y & 0 \\ 0 & I \end{bmatrix} x$$

is more involved. As before, because Y is nonsingular

$$\begin{bmatrix} B_u^T & D_{zu}^T \end{bmatrix} y = 0, \qquad\qquad y = \begin{bmatrix} Y & 0 \\ 0 & I \end{bmatrix} x \neq 0.$$

Let

$$\begin{bmatrix} E_1 \\ E_2 \end{bmatrix} := \begin{bmatrix} B_u^T & D_{zu}^T \end{bmatrix}^\perp$$

so that

$$\mathcal{B}^\perp = \begin{bmatrix} Y^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} B_u^T & D_{zu}^T \end{bmatrix}^\perp = \begin{bmatrix} Y^{-1} E_1 \\ E_2 \end{bmatrix}.$$

This choice produce the inequalities

$$\begin{aligned}
0 &\succ \mathcal{B}^{\perp T} \mathcal{Q} \mathcal{B}^{\perp} \\
&= \begin{bmatrix} E_1^T Y^{-1} & E_2^T \end{bmatrix} \begin{bmatrix} A^T Y + Y A & C_z^T \\ C_z & -I \end{bmatrix} \begin{bmatrix} Y^{-1} E_1 \\ E_2 \end{bmatrix} \\
&= \begin{bmatrix} E_1^T & E_2^T \end{bmatrix} \begin{bmatrix} Y^{-1} A^T + A Y^{-1} & Y^{-1} C_z^T \\ C_z Y^{-1} & -I \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}.
\end{aligned}$$

and

$$\begin{aligned}
0 &\succ \mathcal{C}^{\perp T} \mathcal{Q} \mathcal{C}^{\perp} \\
&= \begin{bmatrix} 0 & I \end{bmatrix} \begin{bmatrix} A^T Y + Y A & C_z^T \\ C_z & -I \end{bmatrix} \begin{bmatrix} 0 \\ I \end{bmatrix} \\
&= -I
\end{aligned}$$

which is trivially satisfied. The first inequality, along with the cost inequality derived previously can be linearized with the change-of-variables $X := Y^{-1}$.

9.3.2 Summary: LQR revisited (second form)

The optimal state feedback controller

$$u(t) = Kx(t)$$

which can be computed from the solution to the SDP in the variables $X \in \mathbb{S}^n$, $Z \in \mathbb{S}^r$ solve the SDP

$$\begin{aligned}
\min_{X \in \mathbb{S}^n, Z \in \mathbb{S}^r} \quad & \mathrm{trace}(ZW) \\
\text{s.t.} \quad & \begin{bmatrix} E_1^T & E_2^T \end{bmatrix} \begin{bmatrix} AX + XA^T & XC_z^T \\ C_z X & -I \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} \prec 0, \\
& \begin{bmatrix} Z & B_w^T \\ B_w & X \end{bmatrix} \succeq 0, \quad X \succ 0.
\end{aligned}$$

where

$$\begin{bmatrix} E_1 \\ E_2 \end{bmatrix} = \begin{bmatrix} B_u^T & D_{zu}^T \end{bmatrix}^\perp$$

stabilizes the closed loop system and minimizes the cost function

$$J := \lim_{t \to \infty} E\left[z(t)^T z(t)\right]$$

for the LTI system

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + B_u u(t) + B_w w(t), \quad x(0) = 0 \\
z(t) &= C_z x(t) + D_{zu} u(t)
\end{aligned}$$

under the assumptions

a) $D_{zu}^T D_{zu} \succ 0$;

b) (A, B_u) stabilizable;

c) $w(t)$ is a Gaussian white noise vector with zero mean and covariance $W \succ 0$.

9.3.3 Computing $\mathcal{B}^\perp$ under the assumption $D_{zu}^T D_{zu} \succ 0$

Proceed as before but now notice that if $D_{zu}^T D_{zu} \succ 0$ then

$$\begin{bmatrix} E_1 \\ E_2 \end{bmatrix} = \begin{bmatrix} I & 0 \\ -D_{zu}(D_{zu}^T D_{zu})^{-1} B_u^T & (D_{zu}^T)^\perp \end{bmatrix} = \begin{bmatrix} B_u^T & D_{zu}^T \end{bmatrix}^\perp$$

is a possible choice for E_1 and E_2 so that, using $X := Y^{-1}$ we have

$$\begin{aligned}
0 &\succ \mathcal{B}^{\perp T} \mathcal{Q} \mathcal{B}^\perp \\
&= \begin{bmatrix} E_1^T & E_2^T \end{bmatrix} \begin{bmatrix} AY^{-1} + Y^{-1}A^T + & Y^{-1}C_z^T \\ C_z Y^{-1} & -I \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} \\
&= \begin{bmatrix} I & 0 \\ -D_{zu}(D_{zu}^T D_{zu})^{-1} B_u^T & (D_{zu}^T)^\perp \end{bmatrix}^T \\
&\qquad \times \begin{bmatrix} AX + XA^T & XC_z^T \\ C_z X & -I \end{bmatrix} \begin{bmatrix} I & 0 \\ -D_{zu}(D_{zu}^T D_{zu})^{-1} B_u^T & (D_{zu}^T)^\perp \end{bmatrix} \\
&= \begin{bmatrix} I & 0 \\ -D_{zu}(D_{zu}^T D_{zu})^{-1} B_u^T & (D_{zu}^T)^\perp \end{bmatrix}^T \\
&\qquad \times \begin{bmatrix} AX + X\tilde{A}^T & XC_z^T (D_{zu}^T)^\perp \\ C_z X + D_{zu}(D_{zu}^T D_{zu})^{-1} B_u^T & (D_{zu}^T)^\perp \end{bmatrix} \\
&= \begin{bmatrix} \tilde{A}X + X\tilde{A}^T - B_u(D_{zu}^T D_{zu})^{-1} B_u^T & XC_z^T (D_{zu}^T)^\perp \\ (D_{zu}^T)^{\perp T} C_z X & (D_{zu}^T)^{\perp T} (D_{zu}^T)^\perp \end{bmatrix}
\end{aligned}$$

where

$$\tilde{A} := A - B_u (D_{zu}^T D_{zu})^{-1} D_{zu}^T C_z.$$

9.3.4 Summary: LQR revisited (third form)

The optimal state feedback controller

$$u(t) = Kx(t)$$

which can be computed from the solution to the SDP in the variables $X \in \mathbb{S}^n$, $Z \in \mathbb{S}^r$ solve the SDP

$$\begin{aligned} \min_{X \in \mathbb{S}^n, Z \in \mathbb{S}^r} \quad & \mathrm{trace}(ZW) \\ \text{s.t.} \quad & \begin{bmatrix} \tilde{A}X + X\tilde{A}^T - B_u(D_{zu}^T D_{zu})^{-1}B_u^T & XC_z^T(D_{zu}^T)^{\perp} \\ (D_{zu}^T)^{\perp T} C_z X & (D_{zu}^T)^{\perp T}(D_{zu}^T)^{\perp} \end{bmatrix} \prec 0, \\ & \begin{bmatrix} Z & B_w^T \\ B_w & X \end{bmatrix} \succeq 0, \quad X \succ 0. \end{aligned}$$

where

$$\tilde{A} := A - B_u(D_{zu}^T D_{zu})^{-1} D_{zu}^T C_z,$$

stabilizes the closed loop system and minimizes the cost function

$$J := \lim_{t \to \infty} E\left[z(t)^T z(t)\right]$$

for the LTI system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_u u(t) + B_w w(t), \quad x(0) = 0 \\ z(t) &= C_z x(t) + D_{zu} u(t) \end{aligned}$$

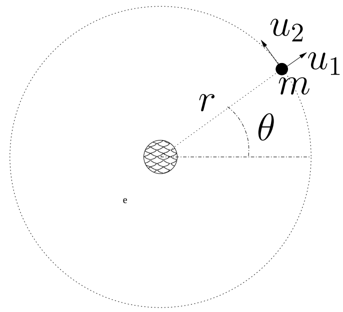
under the assumptions

a) $D_{zu}^T D_{zu} \succ 0$;

b) (A, B_u) stabilizable;

c) $w(t)$ is a Gaussian white noise vector with zero mean and covariance $W \succ 0$.

9.4 Example: satellite in circular orbit



Satellite of mass m with thrust in the radial direction u_1 and in the tangential direction u_2. Continuing...

$$m(\ddot{r} - r\dot{\theta}^2) = u_1 - \frac{km}{r^2} + w_1,$$
$$m(2\dot{r}\dot{\theta} + r\ddot{\theta}) = u_2 + w_2,$$

where w_1 and w_2 are independent white noise disturbances with variances δ_1 and δ_2.
As before, putting in state space and linearizing around

$$\bar{x}(t) = \begin{pmatrix} \bar{r} \\ \bar{\omega} t \\ 0 \\ \bar{\omega} \end{pmatrix}, \qquad \bar{u} = \begin{pmatrix} \bar{u}_1 \\ \bar{u}_2 \end{pmatrix} = 0, \qquad \bar{w} = \begin{pmatrix} \bar{w}_1 \\ \bar{w}_2 \end{pmatrix} = 0, \qquad \bar{\omega} = \sqrt{k/\bar{r}^3}$$

one obtains the linearized system

$$\dot{x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 3\,\bar{\omega}^2 & 0 & 0 & 2\,\bar{r}\bar{\omega} \\ 0 & 0 & -2\,\bar{\omega}/\bar{r} & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1/m & 0 \\ 0 & 1/(m\bar{r}) \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1/m & 0 \\ 0 & 1/(m\bar{r}) \end{bmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.$$

Problem: Given

$$m = 100 \text{ kg}, \qquad \bar{r} = R + 300 \text{ km}, \qquad \bar{k} = GM$$

where $G \approx 6.673 \times 10^{-11}$ N m^2/kg^2 is the universal gravitational constant, and $M \approx 5.98 \times 10^{24}$ kg and $R \approx 6.37 \times 10^3$ km are the mass and radius of the earth. If the variances δ_1 and δ_2 are $0.1N$ find solutions to the LQR control problem where

$$Q = I, \qquad R = \rho I,$$

using u_2 only first, then using u_1 and u_2.

```
% MAE 280 B - Linear Control Design
% Mauricio de Oliveira
%
% LMI LQR Control
%
m = 100;               % 100 kg
r = 300E3;             % 300 km
R = 6.37E6;            % 6.37 10^3 km
G = 6.673E-11;         % 6.673 N m^2/kg^2
M = 5.98E24;           % 5.98 10^24 kg
k = G * M;             % gravitational force constant
w = sqrt(k/((R+r)^3)); % angular velocity (rad/s)
v = w * (R + r);       % "ground" velocity (m/s)

% linearized system matrices
A = [0 0 1 0; 0 0 0 1; 3*w^2 0 0 2*(r+R)*w; 0 0 -2*w/(r+R) 0];
Bu = [0 0; 0 0; 1/m 0; 0 1/(m*r)];
Bw = [0 0; 0 0; 1/m 0; 0 1/(m*r)];

% noise variances
W = 0.1 * eye(2)
W =
   1.0000e-01            0
            0   1.0000e-01

% scale
T = diag([1 r 1 r])
T =
           1           0           0           0
           0      300000           0           0
           0           0           1           0
           0           0           0      300000

% similarity transformation
At = T * A / T
At =
            0            0   1.0000e+00            0
            0            0            0   1.0000e+00
   4.0343e-06            0            0   5.1565e-02
            0            0  -1.0432e-04            0
But = T * Bu
But =
            0            0
            0            0
   1.0000e-02            0
            0   1.0000e-02
Bwt = T * Bw
Bwt =
            0            0
            0            0
   1.0000e-02            0
            0   1.0000e-02
```

```
% optimal LQR state feedback control (using u2 only)
n = size(At,1);
m = 1;

% LQR solution
Cz = [eye(n); zeros(m,n)];
Dzu = [zeros(n,m); eye(m)];

[K,X,S] = lqr(At, But(:,2), Cz' * Cz, Dzu' * Dzu);
Klqr = - K
Klqr =
  -1.0055e+00   1.0000e+00  -5.2522e+01  -1.8511e+01

% LMI solution
% declare variables
X = sdpvar(n,n,'symmetric');
Z = sdpvar(m,m,'symmetric');
L = sdpvar(m,n);

% declare LMIs
LMI1 = [At*X+X*At'+But(:,2)*L+L'*But(:,2)', X*Cz'+L'*Dzu';...
        Cz*X+Dzu*L, -eye(m+n)];
LMI2 = [Z, Bwt(:,2)'; Bwt(:,2), X];
LMI3 = X;

LMI = set(LMI1 < 0) + set(LMI2 > 0) + set(LMI3 > 0);
options = sdpsettings('solver','sedumi');
solution =solvesdp(LMI,trace(W(2,2)*Z),options)
SeDuMi 1.1R3 by AdvOL, 2006 and Jos F. Sturm, 1998-2003.
Alg = 2: xz-corrector, theta = 0.250, beta = 0.500
eqs m = 15, order n = 19, dim = 123, blocks = 4
nnz(A) = 65 + 0, nnz(ADA) = 217, nnz(L) = 116
 it :     b*y       gap    delta  rate   t/tP*  t/tD*   feas cg cg  prec
  0 :            4.49E+00 0.000
  1 :  -6.05E-02 1.02E+00 0.000 0.2270 0.9000 0.9000   2.18  1  1  1.4E+00
  2 :  -1.66E-02 2.34E-01 0.000 0.2298 0.9000 0.9000   1.64  1  1  6.1E-01
  3 :  -3.52E-03 5.82E-02 0.000 0.2484 0.9000 0.9000   1.25  1  1  4.6E-01
  4 :  -1.59E-03 1.67E-02 0.000 0.2862 0.9000 0.9000   1.14  1  1  4.5E-01
  5 :  -2.29E-03 5.42E-03 0.000 0.3256 0.9000 0.9000   0.86  1  1  5.6E-01
  6 :  -3.45E-03 2.22E-03 0.000 0.4094 0.9000 0.9000   0.51  1  1  2.5E-01
  7 :  -4.71E-03 9.46E-04 0.000 0.4262 0.9000 0.9000   0.45  1  1  1.4E-01
  8 :  -5.65E-03 4.91E-04 0.000 0.5188 0.9000 0.9000   0.32  1  1  1.1E-01
  9 :  -7.25E-03 2.24E-04 0.000 0.4556 0.9000 0.9000   0.43  1  1  5.3E-02
 10 :  -8.33E-03 1.05E-04 0.000 0.4707 0.9000 0.9000   0.12  1  1  1.7E-02
 11 :  -1.07E-02 3.89E-05 0.000 0.3694 0.9000 0.9000   0.40  1  1  1.7E-04
 12 :  -1.23E-02 1.50E-05 0.000 0.3863 0.9000 0.9000   0.26  1  1  1.1E-04
 13 :  -1.40E-02 5.31E-06 0.000 0.3537 0.9000 0.9000   0.43  1  1  5.3E-05
 14 :  -1.52E-02 2.11E-06 0.000 0.3977 0.9000 0.9000   0.33  1  1  3.2E-05
 15 :  -1.63E-02 7.42E-07 0.000 0.3513 0.9000 0.9000   0.46  1  1  1.5E-05
 16 :  -1.70E-02 2.94E-07 0.000 0.3962 0.9000 0.9000   0.37  1  1  8.6E-06
 17 :  -1.76E-02 1.04E-07 0.000 0.3531 0.9000 0.9000   0.49  1  1  4.0E-06
 18 :  -1.79E-02 4.26E-08 0.000 0.4099 0.9000 0.9000   0.46  1  1  2.2E-06
 19 :  -1.82E-02 1.41E-08 0.000 0.3322 0.9000 0.9000   0.62  1  1  8.9E-07
 20 :  -1.84E-02 5.15E-09 0.000 0.3645 0.9000 0.9000   0.63  1  1  4.0E-07
```

```
 21 :  -1.84E-02 1.57E-09 0.000 0.3048 0.9000 0.9000   0.77  1  1  1.4E-07
 22 :  -1.85E-02 4.67E-10 0.000 0.2971 0.9000 0.9000   0.79  1  1  4.7E-08
 23 :  -1.85E-02 1.58E-10 0.000 0.3389 0.9000 0.9000   0.87  2  2  1.7E-08
 24 :  -1.85E-02 7.56E-11 0.000 0.4781 0.9000 0.9000   0.80  2  2  9.3E-09
 25 :  -1.85E-02 2.46E-11 0.000 0.3253 0.9000 0.9000   0.80  2  2  3.4E-09
 26 :  -1.85E-02 1.20E-11 0.000 0.4868 0.9000 0.9000   0.57  2  2  2.2E-09
 27 :  -1.85E-02 5.28E-12 0.000 0.4406 0.9000 0.9000   0.48  2  2  1.3E-09
 28 :  -1.85E-02 2.81E-12 0.000 0.5320 0.9000 0.9000   0.26  2  2  9.6E-10

iter seconds digits       c*x               b*y
 28      0.3   Inf -1.8504781972e-02 -1.8504688921e-02
|Ax-b| =   8.8e-10, [Ay-c]_+ =   3.0E-10, |x|=  5.0e+03, |y|=  2.4e-01

Detailed timing (sec)
   Pre          IPM          Post
1.500E-01    3.400E-01    5.000E-02
Max-norms: ||b||=1.000000e-01, ||c|| = 1,
Cholesky |add|=0, |skip| = 0, ||L.L|| = 1354.19.
solution =
    yalmiptime: 6.0114e-01
    solvertime: 6.8049e-01
          info: 'Numerical problems (SeDuMi-1.1)'
       problem: 4
        dimacs: [7.9553e-10 0 0 1.4806e-10 -8.9730e-08 -8.9634e-08]

% Construct K and check closed loop stability
Klqr
Klqr =
  -1.0055e+00   1.0000e+00  -5.2522e+01  -1.8511e+01
K = double(L) / double(X)
K =
  -1.0035e+00   2.1960e-01  -3.7365e+01  -1.8505e+01
Acl = At + But(:,2)*K;
eig(Acl)
ans =
  -6.7474e-02 + 7.5883e-02i
  -6.7474e-02 - 7.5883e-02i
  -5.0082e-02
  -1.7156e-05

% optimal LQR state feedback control (using u1 and u2)
n = size(At,1);
m = 2;

% LQR solution
Cz = [eye(n); zeros(m,n)];
Dzu = [zeros(n,m); eye(m)];

[K,X,S] = lqr(At, But, Cz' * Cz, Dzu' * Dzu);
Klqr = - K
Klqr =
  -9.8444e-01   1.7795e-01  -1.3843e+01  -2.4919e+00
  -1.7795e-01  -9.8404e-01  -2.4919e+00  -1.4741e+01
```

```
% LMI solution
% declare variables
X = sdpvar(n,n,'symmetric');
Z = sdpvar(m,m,'symmetric');
L = sdpvar(m,n);

% declare LMIs
LMI1 = [At*X+X*At'+But*L+L'*But', X*Cz'+L'*Dzu';...
        Cz*X+Dzu*L, -eye(m+n)];
LMI2 = [Z, Bwt'; Bwt, X];

LMI = set(LMI1 < 0) + set(LMI2 > 0);
options = sdpsettings('solver','sedumi');
solution =solvesdp(LMI,trace(W*Z),options)
SeDuMi 1.1R3 by AdvOL, 2006 and Jos F. Sturm, 1998-2003.
Alg = 2: xz-corrector, theta = 0.250, beta = 0.500
eqs m = 21, order n = 17, dim = 137, blocks = 3
nnz(A) = 65 + 0, nnz(ADA) = 393, nnz(L) = 207
 it :     b*y       gap    delta  rate   t/tP*  t/tD*   feas cg cg  prec
  0 :            5.86E+00 0.000
  1 :  -1.21E-01 1.27E+00 0.000 0.2164 0.9000 0.9000   2.32  1  1  1.4E+00
  2 :  -3.20E-02 2.53E-01 0.000 0.1998 0.9000 0.9000   1.54  1  1  7.0E-01
  3 :  -5.95E-03 5.32E-02 0.000 0.2099 0.9000 0.9000   1.23  1  1  5.4E-01
  4 :  -3.86E-03 1.48E-02 0.000 0.2792 0.9000 0.9000   1.03  1  1  6.7E-01
  5 :  -5.46E-03 4.42E-03 0.000 0.2976 0.9000 0.9000   0.53  1  1  2.6E-01
  6 :  -7.75E-03 1.53E-03 0.000 0.3472 0.9000 0.9000   0.14  1  1  1.0E-01
  7 :  -1.09E-02 5.40E-04 0.000 0.3522 0.9000 0.9000   0.17  1  1  8.2E-03
  8 :  -1.38E-02 2.13E-04 0.000 0.3942 0.9000 0.9000   0.12  1  1  5.6E-04
  9 :  -1.74E-02 7.85E-05 0.000 0.3685 0.9000 0.9000   0.22  1  1  3.1E-04
 10 :  -2.02E-02 3.39E-05 0.000 0.4321 0.9000 0.9000   0.17  1  1  2.1E-04
 11 :  -2.38E-02 1.12E-05 0.000 0.3317 0.9000 0.9000   0.38  1  1  9.4E-05
 12 :  -2.60E-02 4.43E-06 0.000 0.3936 0.9000 0.9000   0.45  1  1  5.0E-05
 13 :  -2.78E-02 1.09E-06 0.000 0.2468 0.9000 0.9000   0.70  1  1  1.4E-05
 14 :  -2.84E-02 2.39E-07 0.000 0.2183 0.9000 0.9000   0.85  1  1  3.4E-06
 15 :  -2.86E-02 9.96E-09 0.000 0.0418 0.9900 0.9900   0.96  1  1  1.4E-07
 16 :  -2.86E-02 1.00E-10 0.154 0.0101 0.9990 0.9990   0.99  1  1  1.5E-09
 17 :  -2.86E-02 4.47E-12 0.067 0.0446 0.9900 0.9900   1.00  1  1  6.5E-11

iter seconds digits       c*x               b*y
 17      0.1   Inf -2.8583945115e-02 -2.8583944835e-02
|Ax-b| =   4.4e-11, [Ay-c]_+ =   2.0E-11, |x|=  1.6e+02, |y|=  2.9e-01

Detailed timing (sec)
   Pre          IPM          Post
1.000E-02    9.000E-02    1.000E-02
Max-norms: ||b||=1.000000e-01, ||c|| = 1,
Cholesky |add|=0, |skip| = 0, ||L.L|| = 218.956.
solution =
    yalmiptime: 3.8412e-02
    solvertime: 1.1421e-01
          info: 'No problems detected (SeDuMi-1.1)'
       problem: 0
        dimacs: [3.9717e-11 0 0 9.9449e-12 -2.6457e-10 -2.5799e-10]
```

```
% Construct K and check closed loop stability
Klqr
Klqr =
  -9.8444e-01   1.7795e-01  -1.3843e+01  -2.4919e+00
  -1.7795e-01  -9.8404e-01  -2.4919e+00  -1.4741e+01
K = double(L) / double(X)
K =
  -9.8444e-01   1.7795e-01  -1.3843e+01  -2.4919e+00
  -1.7795e-01  -9.8404e-01  -2.4919e+00  -1.4741e+01
Acl = At + But*K;
eig(Acl)
ans =
  -7.1524e-02 + 8.2854e-02i
  -7.1524e-02 - 8.2854e-02i
  -7.1395e-02 + 5.7005e-02i
  -7.1395e-02 - 5.7005e-02i

diary off
```

Let's solve it again with the second form.

```
% MAE 280 B - Linear Control Design
% Mauricio de Oliveira
%
% LMI LQR Control
%
m = 100;              % 100 kg
r = 300E3;            % 300 km
R = 6.37E6;           % 6.37 10^3 km
G = 6.673E-11;        % 6.673 N m^2/kg^2
M = 5.98E24;          % 5.98 10^24 kg
k = G * M;            % gravitational force constant
w = sqrt(k/((R+r)^3)); % angular velocity (rad/s)
v = w * (R + r);      % "ground" velocity (m/s)

% linearized system matrices
A = [0 0 1 0; 0 0 0 1; 3*w^2 0 0 2*(r+R)*w; 0 0 -2*w/(r+R) 0];
Bu = [0 0; 0 0; 1/m 0; 0 1/(m*r)];
Bw = [0 0; 0 0; 1/m 0; 0 1/(m*r)];

% noise variances
W = 0.1 * eye(2)
W =
   1.0000e-01            0
            0   1.0000e-01

% scale
T = diag([1 r 1 r])
T =
           1           0           0           0
           0      300000           0           0
           0           0           1           0
           0           0           0      300000

% similarity transformation
At = T * A / T
At =
            0            0   1.0000e+00            0
            0            0            0   1.0000e+00
   4.0343e-06            0            0   5.1565e-02
            0            0  -1.0432e-04            0
But = T * Bu
But =
            0            0
            0            0
   1.0000e-02            0
            0   1.0000e-02
Bwt = T * Bw
Bwt =
            0            0
            0            0
   1.0000e-02            0
            0   1.0000e-02
```

```
% optimal LQR state feedback control (using u1 and u2)
n = size(At,1);
m = size(But,2);

% LQR solution
Cz = [eye(n); zeros(m,n)];
Dzu = [zeros(n,m); eye(m)];

[K,X,S] = lqr(At, But, Cz' * Cz, Dzu' * Dzu);
Klqr = - K
Klqr =
  -9.8444e-01   1.7795e-01  -1.3843e+01  -2.4919e+00
  -1.7795e-01  -9.8404e-01  -2.4919e+00  -1.4741e+01

% LMI solution (second form)
E = null([But' Dzu']);

% declare variables
X = sdpvar(n,n,'symmetric');
Z = sdpvar(m,m,'symmetric');

% declare LMIs
LMI1 = E'*[At*X+X*At', X*Cz';...
        Cz*X, -eye(m+n)]*E;
LMI2 = [Z, Bwt'; Bwt, X];

LMI = set(LMI1 < 0) + set(LMI2 > 0);
options = sdpsettings('solver','sedumi');
solution =solvesdp(LMI,trace(W*Z),options)
SeDuMi 1.1R3 by AdvOL, 2006 and Jos F. Sturm, 1998-2003.
Alg = 2: xz-corrector, theta = 0.250, beta = 0.500
eqs m = 13, order n = 15, dim = 101, blocks = 3
nnz(A) = 104 + 0, nnz(ADA) = 169, nnz(L) = 91
 it :     b*y       gap    delta  rate   t/tP*  t/tD*   feas cg cg  prec
  0 :            4.81E+00 0.000
  1 :  -1.24E-01 1.05E+00 0.000 0.2188 0.9000 0.9000   2.16  1  1  1.5E+00
  2 :  -2.72E-02 2.21E-01 0.000 0.2096 0.9000 0.9000   1.72  1  1  6.9E-01
  3 :  -4.83E-03 4.56E-02 0.000 0.2065 0.9000 0.9000   1.28  1  1  5.2E-01
  4 :  -3.67E-03 1.25E-02 0.000 0.2735 0.9000 0.9000   1.02  1  1  5.0E-01
  5 :  -5.44E-03 3.61E-03 0.000 0.2894 0.9000 0.9000   0.42  1  1  2.0E-01
  6 :  -7.84E-03 1.16E-03 0.000 0.3227 0.9000 0.9000   0.10  1  1  5.9E-02
  7 :  -1.09E-02 4.06E-04 0.000 0.3486 0.9000 0.9000   0.14  1  1  8.7E-04
  8 :  -1.39E-02 1.57E-04 0.000 0.3876 0.9000 0.9000   0.09  1  1  5.7E-04
  9 :  -1.76E-02 5.67E-05 0.000 0.3607 0.9000 0.9000   0.21  1  1  3.1E-04
 10 :  -2.05E-02 2.34E-05 0.000 0.4130 0.9000 0.9000   0.16  1  1  2.1E-04
 11 :  -2.42E-02 7.55E-06 0.000 0.3222 0.9000 0.9000   0.38  1  1  9.0E-05
 12 :  -2.63E-02 2.83E-06 0.000 0.3752 0.9000 0.9000   0.47  1  1  4.5E-05
 13 :  -2.79E-02 6.59E-07 0.000 0.2328 0.9000 0.9000   0.72  1  1  1.2E-05
 14 :  -2.85E-02 1.25E-07 0.000 0.1891 0.9000 0.9000   0.87  1  1  2.4E-06
 15 :  -2.86E-02 4.64E-09 0.000 0.0372 0.9900 0.9900   0.97  1  1  9.2E-08
 16 :  -2.86E-02 4.82E-11 0.167 0.0104 0.9990 0.9990   0.99  1  1  9.6E-10

iter seconds digits       c*x               b*y
```

```
 16      0.2   Inf -2.8583895204e-02 -2.8583889484e-02
|Ax-b| =   6.5e-10, [Ay-c]_+ =   3.1E-10, |x|=  1.6e+02, |y|=  2.9e-01

Detailed timing (sec)
   Pre          IPM          Post
1.400E-01    2.400E-01    4.000E-02
Max-norms: ||b||=1.000000e-01, ||c|| = 1,
Cholesky |add|=0, |skip| = 0, ||L.L|| = 235.364.
solution =
    yalmiptime: 6.2339e-01
    solvertime: 4.9829e-01
          info: 'No problems detected (SeDuMi-1.1)'
       problem: 0
        dimacs: [5.9141e-10 0 0 1.5708e-10 -5.4113e-09 -5.3135e-09]

% Construct K and check closed loop stability
Y = inv(double(X));
QQ = [Y*At+At'*Y, Cz';...
     Cz, -eye(m+n)];
BB = [But'*Y Dzu'];
CC = [eye(n) zeros(n,m+n)];

mu = 1e8
mu =
   100000000
max(eig(QQ - mu*BB'*BB))
ans =
  -2.8505e+10
max(eig(QQ - mu*CC'*CC))
ans =
  -1.0000e+00

Phi = QQ - mu*BB'*BB - mu*CC'*CC;
XX = mu^2 * CC * inv(Phi) * BB';

% Check closed loop stability
Klqr
Klqr =
  -9.8444e-01   1.7795e-01  -1.3843e+01  -2.4919e+00
  -1.7795e-01  -9.8404e-01  -2.4919e+00  -1.4741e+01
Klmi = XX'
Klmi =
  -9.8444e-01   1.7795e-01  -1.3843e+01  -2.4919e+00
  -1.7795e-01  -9.8404e-01  -2.4919e+00  -1.4741e+01
Acl = At + But*Klmi;
eig(Acl)
ans =
  -7.1524e-02 + 8.2854e-02i
  -7.1524e-02 - 8.2854e-02i
  -7.1395e-02 + 5.7005e-02i
  -7.1395e-02 - 5.7005e-02i

diary off
```