7 Stability

7.1 Linear Systems Stability

Consider the Continuous-Time (CT) Linear Time-Invariant (LTI) system

$$\dot{x}(t) = A\,x(t), \qquad x(0) = x_0, \qquad A \in \mathbb{R}^{n \times n}, \quad x_0 \in \mathbb{R}^n. \tag{14}$$

The origin $x = 0$ is a *globally asymptotically stable equilibrium* point of system (14) if

$$\lim_{t \to \infty} x(t) = 0, \quad \text{for all } \; x_0 \neq 0.$$

Lyapunov's second method. Consider a differentiable function $V : \mathbb{R}^n \to \mathbb{R}$

a) $V(x) \geq 0$ for all $x \in \mathbb{R}^n$;

b) $V(x) = 0$ iff $x = 0$.

If

$$\dot{V}(x(t)) < 0, \qquad \text{for all } x(t) \text{ satisfying (14) with } x_0 \neq 0$$

then $x = 0$ is a globally asymptotically stable equilibrium point of system (14).

Any such V is a *Lyapunov function*.

Theorem [Lyapunov]: The following statements regarding system (14) are equivalent:

a) $x = 0$ is a globally asymptotically stable equilibrium;

b) there exists a *quadratic Lyapunov function* $V(x) = x^T P x$, $P \in \mathbb{R}^{n \times n}$.

7.2 Lyapunov Stability Test

Problem: Given system (14), find if there exists a matrix $P \in \mathbb{R}^{n\times n}$ such that

a) $V(x) = x^T P x > 0, \quad$ for all $x \neq 0$

b) $\dot{V}(x) < 0, \quad$ for all $\dot{x} = Ax, \quad x \neq 0.$

Remarks: only the "symmetric part" of P matters:
First

$$V(x) = x^T P x = \frac{1}{2} x^T \left(P + P^T\right) x > 0, \text{for all } x \neq 0 \quad \iff \quad P + P^T \succ 0.$$

Second

$$\dot{V}(x) = \langle \nabla V(x), \dot{x} \rangle$$

In order to compute ∇f we use (Gateaux differential)

$$\langle \nabla f, h \rangle = \lim_{\epsilon \to 0} \frac{f(x + \epsilon h) - f(x)}{\epsilon}$$

For $V = x^T P x$

$$\begin{aligned} V(x + \epsilon h) &= (x + \epsilon h)^T P (x + \epsilon h), \\ &= x^T P x + \epsilon \left(h^T P x + x^T P h\right) + \mathcal{O}(\epsilon^2), \\ &= V(x) + \epsilon \left\langle \left(P + P^T\right) x, h \right\rangle + \mathcal{O}(\epsilon^2) \end{aligned}$$

hence

$$\lim_{\epsilon \to 0} \frac{V(x + \epsilon h) - V(x)}{\epsilon} = \left\langle \left(P + P^T\right) x, h \right\rangle \quad \Longrightarrow \quad \nabla V(x) = \left(P + P^T\right) x,$$

so one can assume $P \in \mathbb{S}^n$ without loss of generality.

Problem: Given system (14), find if there exists a matrix $P \in \mathbb{S}^n$ such that

a) $V(x) = x^T P x > 0, \quad$ for all $x \neq 0$

b) $\dot{V}(x) < 0, \quad$ for all $\dot{x} = Ax, \quad x \neq 0.$

Solution:
Condition a)

$$V(x) = x^T P x > 0, \quad \text{for all } x \neq 0 \qquad \iff \qquad P \succ 0.$$

Condition b) For all $x \neq 0$

$$\begin{aligned} 0 > \dot{V}(x) &= \langle \nabla V(x), \dot{x} \rangle \\ &= 2 \langle Px, Ax \rangle \\ &= 2x^T P A x \\ &= x^T (A^T P + PA) x \end{aligned}$$

which is equivalent to

$$A^T P + PA \prec 0.$$

Lyapunov Stability Test: Given system (14), find if there exists a matrix $P \in \mathbb{S}^n$ such that the LMI

$$P \succ 0, \qquad A^T P + PA \prec 0,$$

is feasible.

7.3 Relation with Lyapunov Equations

Lyapunov's First Method. Consider the general nonlinear system

$$\dot{x} = f(x), \tag{15}$$

and the linearized system at the equilibrium point

$$\dot{x} = Ax, \qquad\qquad A = \nabla^T f(0).$$

Theorem [Lyapunov]: The origin $x = 0$ is a locally asymptotically stable equilibrium point of system (15) if and only if the matrix $P \in \mathbb{S}^n$ that solves the *Lyapunov equation*

$$A^T P + PA + Q = 0 \tag{16}$$

is positive definite, i.e., $P \succ 0$, for some matrix $Q \succ 0$.

Remarks:

a) For linear systems, local becomes global.

b) With what we know about Lyapunov equations if $Q \succ 0$ then (A, Q) is observable and $P \succ 0$ if and only if A is Hurwitz.

c) Furthermore, if A is Hurwitz then $P \succ 0$ for *any* matrix $Q \succ 0$.

7.3.1 From Lyapunov Equation to Lyapunov Inequality

Since $Q \succ 0$ the Lyapunov equation (16) provides $P \succ 0$ and

$$A^T P + PA = -Q \prec 0$$

7.3.2 From Lyapunov Inequality to Lyapunov Equation

If there exists $P \succ 0$ such that

$$A^T P + PA \prec 0$$

then there exists also $Q \succ 0$ such that $A^T P + PA + Q = 0$.

7.4 Discrete-Time Systems

Consider the Discrete-Time (DT) Linear Time-Invariant (LTI) system

$$x(k+1) = Ax(k), \qquad x(0) = x_0, \qquad A \in \mathbb{R}^{n \times n}, \quad x_0 \in \mathbb{R}^n. \tag{17}$$

The origin $x = 0$ is a *globally asymptotically stable equilibrium* point of system (14) if

$$\lim_{k \to \infty} x(k) = 0, \quad \text{for all } x_0 \neq 0.$$

Lyapunov's second method. Consider a differentiable function $V : \mathbb{R}^n \to \mathbb{R}$

a) $V(x) \geq 0$ for all $x \in \mathbb{R}^n$;

b) $V(x) = 0$ iff $x = 0$.

If

$$V(x(k+1)) - V(x(k)) < 0, \qquad \text{for all } x(k) \text{ satisfying (17) with } x_0 \neq 0$$

then $x = 0$ is a globally asymptotically stable equilibrium point of system (17).

Any such V is a *Lyapunov function*.

Theorem [Lyapunov]: The following statements regarding system (17) are equivalent:

a) $x = 0$ is a globally asymptotically stable equilibrium;

b) there exists a *quadratic Lyapunov function* $V(x) = x^T P x$, $P \in \mathbb{S}^n$.

Problem: Given system (17), find if there exists a matrix $P \in \mathbb{S}^n$ such that

a) $V(x) = x^T P x > 0, \quad$ for all $x \neq 0$

b) $V(x(k+1)) - V(x(k)) < 0, \quad$ for all $x(k+1) = Ax(x), \quad x(k) \neq 0.$

Solution:
Condition a). $P \succ 0$.

Condition b). $V(x(k+1)) = x(k+1)^T P x(k+1) = x(k)^T A^T P A x(k)$ so that for all $x(k) \neq 0$

$$\begin{aligned} 0 > V(x(k+1)) - V(x(k)) &= x(k)^T A^T P A x(k) - x(k)^T P x(k) \\ &= x(k)^T \left(A^T P A - P\right) x(k) \end{aligned}$$

which is equivalent to

$$A^T P A - P \prec 0.$$

Lyapunov Stability Test: Given system (17), find if there exists a matrix $P \in \mathbb{S}^n$ such that the LMI

$$P \succ 0, \qquad\qquad A^T P A - P \prec 0,$$

is feasible.

Schur complement provides the equivalent alternative form.

Lyapunov Stability Test: Given system (17), find if there exists a matrix $P \in \mathbb{S}^n$ such that the LMI

$$\begin{bmatrix} P & A^T P \\ PA & P \end{bmatrix} \succ 0,$$

is feasible.

7.4.1 Summary for linear discrete-time systems

The following statements are equivalent:

a) $x = 0$ is a globally asymptotically stable equilibrium of system (17);

b) there exists a *quadratic Lyapunov function* $V(x) = x^T P x$, $P \in \mathbb{S}^n$;

c) there exists a matrix $P \in \mathbb{S}^n$ such that

$$P \succ 0, \qquad\qquad A^T P A - P \prec 0.$$

d) there exists a matrix $P \in \mathbb{S}^n$ such that

$$\begin{bmatrix} P & A^T P \\ PA & P \end{bmatrix} \succ 0,$$

e) matrix $P \in \mathbb{S}^n$ that solves the *Stein equation*

$$A^T P A - P + Q = 0$$

is positive definite, i.e., $P \succ 0$, for some matrix $Q \succ 0$.

f) matrix A is Schur ($\max_i |\lambda_i(A)| < 1$);