5 Optimal Linear Quadratic Gaussian (LQG) Control

Problem:
Given the LTI system

$$\begin{aligned} \dot{x} &= Ax + B_u u + B_w w, \\ y &= C_y x + D_{yw} w, \\ z &= C_z x + D_{zu} u, \end{aligned}$$

and the *observer-based controller*

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + B_u u + F(\hat{y} - y), \\ \hat{y} &= C_y \hat{x}, \\ u &= K\hat{x}. \end{aligned}$$

compute (K, F) that stabilize the closed loop system and minimize the cost function

$$J := \lim_{t \to \infty} E\left[z(t)^T z(t)\right]. \tag{3}$$

Assumptions:

1. (A, B_u) stabilizable,
2. (A, C_y) detectable,
3. $w(t)$ is a Gaussian zero mean white noise with variance $W \succ 0$,
4. $C_z^T D_{zu} = 0$ and $D_{zu}^T D_{zu} \succ 0$,
5. $B_w W D_{yw}^T = 0$ and $D_{yw} W D_{yw}^T \succ 0$.

Remark 1: 4 and 5 can be relaxed.
Remark 2: 4 implies that

$$J = \lim_{t \to \infty} E\left[x(t)^T Q x(t) + u(t)^T R u(t)\right],$$

where

$$Q = C_z^T C_z, \qquad R = D_{zu}^T D_{zu} \succ 0.$$

5.1 The Closed Loop System

The closed loop system is

$$
\begin{aligned}
\begin{pmatrix} \dot{x} \\ \dot{\hat{x}} \end{pmatrix} &= \begin{bmatrix} A & B_u K \\ -FC_y & A + B_u K + FC_y \end{bmatrix} \begin{pmatrix} x \\ \hat{x} \end{pmatrix} + \begin{bmatrix} B_w \\ -FD_{yw} \end{bmatrix} w, \\
z &= \begin{bmatrix} C_z & D_{zu} K \end{bmatrix} \begin{pmatrix} x \\ \hat{x} \end{pmatrix}
\end{aligned}
\tag{4}
$$

As in the estimation problem, we write the dynamics in terms of $e := x - \hat{x}$. Note that

$$
\begin{pmatrix} e \\ \hat{x} \end{pmatrix} = \begin{bmatrix} I & -I \\ 0 & I \end{bmatrix} \begin{pmatrix} x \\ \hat{x} \end{pmatrix}, \qquad \begin{pmatrix} x \\ \hat{x} \end{pmatrix} = \begin{bmatrix} I & I \\ 0 & I \end{bmatrix} \begin{pmatrix} e \\ \hat{x} \end{pmatrix},
$$

and

$$
\begin{aligned}
\begin{pmatrix} \dot{e} \\ \dot{\hat{x}} \end{pmatrix} &= \begin{bmatrix} A + FC_y & 0 \\ -FC_y & A + B_u K \end{bmatrix} \begin{pmatrix} e \\ \hat{x} \end{pmatrix} + \begin{bmatrix} B_w + FD_{yw} \\ -FD_{yw} \end{bmatrix} w, \\
z &= \begin{bmatrix} C_z & C_z + D_{zu} K \end{bmatrix} \begin{pmatrix} e \\ \hat{x} \end{pmatrix}
\end{aligned}
\tag{5}
$$

5.2 Stability of the Closed Loop System

First fantastic result:

The closed loop system (4) is stable
if and only if
K *is a stabilizing state feedback gain and*
F *is a stabilizing state estimation gain.*

Proof: Systems (4) and (5) are **similar**, therefore the closed loop poles are the roots of the polyomial equation

$$
\begin{aligned}
0 &= \det \begin{bmatrix} sI - A & -B_u K \\ FC_y & sI - (A + B_u K + FC_y) \end{bmatrix} \\
&= \det \begin{bmatrix} sI - (A + FC_y) & 0 \\ FC_y & sI - (A + B_u K) \end{bmatrix} \\
&= \det \left[sI - (A + FC_y) \right] \det \left[sI - (A + B_u K) \right]
\end{aligned}
$$

5.3 The Cost Function

Now write the cost function (1) in terms of system (5)

$$J = \lim_{t \to \infty} E\left[z(t)^T z(t)\right].$$

Computing J using the Controllability Gramian of system (5) we have

$$J = \operatorname{trace}\left(\begin{bmatrix} C_z & C_z + D_{zu}K \end{bmatrix} \begin{bmatrix} Y_1 & Y_2 \\ Y_2^T & Y_3 \end{bmatrix} \begin{bmatrix} C_z^T \\ C_z^T + K^T D_{zu}^T \end{bmatrix} \right), \qquad (6)$$

where Y_1, Y_2 and Y_3 satisfy the Lyapunov equation

$$\begin{aligned} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A + FC_y & 0 \\ -FC_y & A + B_u K \end{bmatrix} \begin{bmatrix} Y_1 & Y_2 \\ Y_2^T & Y_3 \end{bmatrix} \\ + \begin{bmatrix} Y_1 & Y_2 \\ Y_2^T & Y_3 \end{bmatrix} \begin{bmatrix} A + FC_y & 0 \\ -FC_y & A + B_u K \end{bmatrix}^T \\ + \begin{bmatrix} B_w + FD_{yw} \\ -FD_{yw} \end{bmatrix} W \begin{bmatrix} (B_w + FD_{yw})^T & -(FD_{yw})^T \end{bmatrix} \end{aligned}$$

Expanding these equations we have

$$\begin{aligned} 0 &= (A + FC_y)Y_1 + Y_1(A + FC_y)^T + (B_w + FD_{yw})W(B_w + FD_{yw})^T, && (7) \\ 0 &= (A + FC_y)Y_2 + Y_2(A + B_u K)^T - Y_1 C_y^T F^T - FD_{yw} W D_{yw}^T F^T, && (8) \\ 0 &= (A + B_u K)Y_3 + Y_3(A + B_u K)^T + FD_{yw} W D_{yw}^Y F^T \\ &\qquad - FC_y Y_2 - Y_2^T C_y^T F^T. && (9) \end{aligned}$$

In (8) we have used the assumption that $B_w W D_{yw}^T = 0$.

5.4 The Optimal Estimator Gain

The closed loop is stable if and only if F is stabilizing. Furthermore

$$J = J_1(Y_1) + J_2(Y_2, Y_3, K)$$

where

$$\begin{aligned} J_1(Y_1) &= \operatorname{trace}(C_z Y_1 C_z^T), \\ J_2(Y_2, Y_3, K) &= \operatorname{trace}([C_z + D_{zu}K] Y_3 [C_z + D_{zu}K]^T) \\ &\qquad + 2\operatorname{trace}(C_z Y_2 [C_z + D_{zu}K]^T). \end{aligned}$$

Note that Y_1 is determined by solving (7)

$$(A + FC_y)Y_1 + Y_1(A + FC_y)^T + (B_w + FD_{yw})W(B_w + FD_{yw})^T = 0,$$

which does not depend on K. Therefore the cost J is minimum if $J_1(Y_1)$ is also minimum. But this is the case if Y_1 is minimum in the sense of the Comparison Lemma[1]. That is, with $Y_1 = Y^*$ satisfying the ARE

$$AY^* + Y^*A^T - Y^*C_y^T(D_{yw}WD_{yw}^T)^{-1}C_yY^* + B_wWB_w^T = 0. \qquad (10)$$

The optimal observer gain is then

$$F^* = -Y^*C_y^T(D_{yw}WD_{yw}^T)^{-1}.$$

5.5 A Temporary Solution For Y_2

Knowing that, we rewrite (8) as

$$0 = AY_2 + Y_2(A + B_uK)^T - Y^*C_y^T(D_{yw}WD_{yw}^T)^{-1}C_yY_2.$$

Note that the above equation is trivially satisfied with $Y_2^* = 0$. Assume this choice of Y_2^* as a matter of faith for the moment.

[1] See section "What if $Y_2 \neq 0$" for a formal proof of a similar fact.

5.6 The Optimal Control Gain

With $Y_1 = Y^*$ and $Y_2 = Y_2^* = 0$, the cost function (20) becomes

$$J = \text{trace}\left[C_z Y^* C_z^T\right] + \text{trace}\left[\left(C_z + D_{zu}K\right) Y_3 \left(C_z + D_{zu}K\right)^T\right].$$

In this form, it is clear that the optimal K is the one that minimizes

$$J_2 = \text{trace}\left[\left(C_z + D_{zu}K\right) Y_3 \left(C_z + D_{zu}K\right)^T\right].$$

subject to equation (9), which for $Y_2^* = 0$ is simply

$$(A + B_u K)Y_3 + Y_3(A + B_u K)^T + Y^* C_y^T (D_{yw} W D_{yw}^T)^{-1} C_y Y^* = 0. \qquad (11)$$

Using our previous duality results, the optimal K is the one that minimizes

$$J_2 = \text{trace}\left[\bar{B}^T X \bar{B}\right], \qquad \bar{B} := Y^* C_y^T (D_{yw} W D_{yw}^T)^{-1/2}$$

where

$$\begin{aligned} 0 &= (A + B_u K)^T X + X(A + B_u K) + \left(C_z + D_{zu}K\right)^T \left(C_z + D_{zu}K\right) \\ &= (A + B_u K)^T X + X(A + B_u K) + C_z^T C_z + K^T D_{zu}^T D_{zu} K. \end{aligned}$$

Using what we know about the LQR problem, the solution is

$$K^* = -(D_{zu}^T D_{zu})^{-1} B_u^T X^*,$$

where X^* satisfies the ARE

$$A^T X^* + X^* A - X^* B_u (D_{zu}^T D_{zu})^{-1} B_u^T X^* + C_z^T C_z = 0.$$

Matrix Y_3^* can be computed from (9) upon substitution of the optimal K^*.

5.7 Summary of LQG Control

Given the LTI system

$$\begin{aligned}\dot{x} &= Ax + B_u u + B_w w,\\ y &= C_y x + D_{yw} w,\\ z &= C_z x + D_{zu} u,\end{aligned}$$

and the *observer-based controller*

$$\begin{aligned}\dot{\hat{x}} &= A\hat{x} + B_u u + F(\hat{y} - y),\\ \hat{y} &= C_y \hat{x},\\ u &= K\hat{x},\end{aligned}$$

if

1. (A, B_u) stabilizable,
2. (A, C_y) detectable,
3. $w(t)$ is a Gaussian zero mean white noise with variance $W \succ 0$,
4. $C_z^T D_{zu} = 0$ and $D_{zu}^T D_{zu} \succ 0$,
5. $B_w W D_{yw}^T = 0$ and $D_{yw} W D_{yw}^T \succ 0$,

then the choice of (K, F) that stabilize the closed loop system and minimize the cost function

$$J := \lim_{t\to\infty} E\left[z(t)^T z(t)\right].$$

is given by

$$K^* = -(D_{zu}^T D_{zu})^{-1} B_u^T X^*,$$

where X^* satisfies the ARE

$$A^T X^* + X^* A - X^* B_u (D_{zu}^T D_{zu})^{-1} B_u^T X^* + C_z^T C_z = 0,$$

and

$$F^* = -Y^* C_y^T (D_{yw} W D_{yw}^T)^{-1},$$

where Y^* satisfies the ARE

$$AY^* + Y^* A^T - Y^* C_y^T (D_{yw} W D_{yw}^T)^{-1} C_y Y^* + B_w W B_w^T = 0.$$

5.8 What if $Y_2 \neq 0$?

The optimal covariance matrix has the form

$$\begin{bmatrix} Y_1^* & Y_2^* \\ Y_2^{*T} & Y_3^* \end{bmatrix} = \begin{bmatrix} Y^* & 0 \\ 0 & Y_3^* \end{bmatrix} \succ 0$$

We claim that this matrix is minimal in the sense of the comparison lemma.
Proof: Assume it is not. Then there exists

$$\begin{bmatrix} Y^* & 0 \\ 0 & Y_3^* \end{bmatrix} \succeq \begin{bmatrix} Y_1 & Y_2 \\ Y_2^T & Y_3 \end{bmatrix} \succ 0$$

This implies that

$$\begin{bmatrix} Y^* - Y_1 & -Y_2 \\ -Y_2^T & Y_3^* - Y_3 \end{bmatrix} \succeq 0 \tag{12}$$

Multiplying (12) on the left by $\begin{bmatrix} I & 0 \end{bmatrix}$ and on the right by $\begin{bmatrix} I \\ 0 \end{bmatrix}$ we obtain that

$$Y^* \succeq Y_1.$$

But since we have already proved that Y^* is minimal

$$Y_1 \succeq Y^*.$$

Therefore, $Y_1 = Y^*$. Using this fact (12) becomes

$$\begin{bmatrix} 0 & -Y_2 \\ -Y_2^T & Y_3^* - Y_3 \end{bmatrix} \succeq 0.$$

Multiplying (12) on the left by $\begin{bmatrix} 0 & I \end{bmatrix}$ and on the right by $\begin{bmatrix} 0 \\ I \end{bmatrix}$ we conclude that

$$Y_3^* \succeq Y_3.$$

Now, matrix (12) is positive semidefinite if and only if

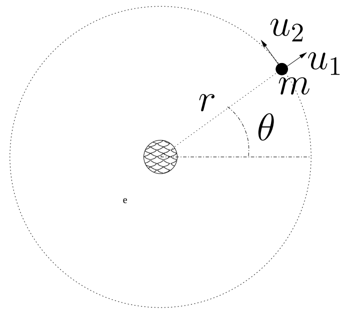
$$\begin{pmatrix} x \\ y \end{pmatrix}^T \begin{bmatrix} 0 & -Y_2 \\ -Y_2^T & Y_3^* - Y_3 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = -2x^T Y_2 y + y^T (Y_3^* - Y_3) y \geq 0, \quad \forall x, y$$

Because, $Y_3^* - Y_3 \succeq 0$, this is certainly true for $Y_2 = 0$. On the other hand, if $Y_2 \neq 0$, all vectors in the form $x = \alpha Y_2 y$ are such that

$$y^T (Y_3^* - Y_3) y - 2\alpha y^T Y_2^T Y_2 y \geq 0, \quad \forall y.$$

As $Y_2^T Y_2 \succeq 0$, for any y such that $Y_2 y \neq 0$ there is a sufficiently big $\alpha > 0$ such that the above inequality is violated, proving that Y_2 must indeed be zero.

5.9 Example: controlling a satellite in circular orbit



Satellite of mass m with thrust in the radial direction u_1 and in the tangential direction u_2. Continuing...

$$m(\ddot{r} - r\dot{\theta}^2) = u_1 - \frac{km}{r^2} + w_1,$$
$$m(2\dot{r}\dot{\theta} + r\ddot{\theta}) = u_2 + w_2,$$

where w_1 and w_2 are independent white noise disturbances with variances δ_1 and δ_2.
As before, putting in state space and linearizing

$$\dot{x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 3\,\bar{\omega}^2 & 0 & 0 & 2\,\bar{r}\bar{\omega} \\ 0 & 0 & -2\,\bar{\omega}/\bar{r} & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1/m & 0 \\ 0 & 1/(m\bar{r}) \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1/m & 0 \\ 0 & 1/(m\bar{r}) \end{bmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

Consider that you have a noisy measurement of θ (x_2)

$$y = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} + v$$

where $Evv^T = \delta_3$.
Problem: Given

$$m = 100 \text{ kg}, \qquad \bar{r} = R + 300 \text{ km}, \qquad \bar{k} = GM$$

where $G \approx 6.673 \times 10^{-11}$ N m^2/kg^2 is the universal gravitational constant, and $M \approx 5.98 \times 10^{24}$ kg and $R \approx 6.37 \times 10^3$ km are the mass and radius of the earth. If the variances $\delta_1 = \delta_2 = \delta_3 = 0.1N$ find solutions to the LQR control problem where

$$Q = I, \qquad R = I,$$

using u_2 only first, then using u_1 and u_2.

```
% MAE 280 B - Linear Control Design
% Mauricio de Oliveira
%
% LQG Control - Part II
%
m = 100;               % 100 kg
r = 300E3;             % 300 km
R = 6.37E6;            % 6.37 10^3 km
G = 6.673E-11;         % 6.673 N m^2/kg^2
M = 5.98E24;           % 5.98 10^24 kg
k = G * M;             % gravitational force constant
w = sqrt(k/((R+r)^3)); % angular velocity (rad/s)
v = w * (R + r);       % "ground" velocity (m/s)

% linearized system matrices
A = [0 0 1 0; 0 0 0 1; 3*w^2 0 0 2*(r+R)*w; 0 0 -2*w/(r+R) 0];
Bu = [0 0; 0 0; 1/m 0; 0 1/(m*r)];
Bw = [0 0; 0 0; 1/m 0; 0 1/(m*r)];

% noise variances
W = 0.1 * eye(2)
W =
   1.0000e-01            0
            0   1.0000e-01

% scale
T = diag([1 r 1 r])
T =
           1           0           0           0
           0      300000           0           0
           0           0           1           0
           0           0           0      300000

% similarity transformation
At = T * A / T
At =
            0            0   1.0000e+00            0
            0            0            0   1.0000e+00
   4.0343e-06            0            0   5.1565e-02
            0            0  -1.0432e-04            0
But = T * Bu
But =
            0            0
            0            0
   1.0000e-02            0
            0   1.0000e-02
Bwt = T * Bw
Bwt =
            0            0
            0            0
   1.0000e-02            0
            0   1.0000e-02
```

```
% measuring x2 (theta)
Cy = [0 1 0 0];
Cyt = Cy / T
Cyt =
            0   3.3333e-06            0            0

% augment noise matrices
Bwt = [Bwt zeros(4,1)]
Bwt =
            0            0            0
            0            0            0
   1.0000e-02            0            0
            0   1.0000e-02            0
Dywt = [zeros(1,2) 1]
Dywt =
     0     0     1

Ww = W
Ww =
   1.0000e-01            0
            0   1.0000e-01
Wv = 0.1
Wv =
   1.0000e-01

Wt = [Ww zeros(2, 1); zeros(1, 2) Wv]
Wt =
   1.0000e-01            0            0
            0   1.0000e-01            0
            0            0   1.0000e-01

% optimal state feedback control (using u2 only)
rho = 1
rho =
     1
Czt = [eye(4); zeros(1,4)];
Dzut = sqrt(rho)*[zeros(4,1); eye(1)];
Czt'*Dzut
ans =
     0
     0
     0
     0
[K,X,S] = lqr(At, But(:,2), Czt'*Czt, Dzut'*Dzut);
K = - K
K =
  -1.0055e+00   1.0000e+00  -5.2522e+01  -1.8511e+01

Acl = At + But(:,2)*K;
eig(Acl)
ans =
  -6.7473e-02 + 7.5883e-02i
  -6.7473e-02 - 7.5883e-02i
  -5.0086e-02
```

```
  -7.8118e-05

% optimal state estimation
[F,X,S] = lqr(At', Cyt', Bwt * Wt * Bwt', Dywt * Wt * Dywt');
F = - F'
F =
   5.7737e+02
  -1.7360e+02
  -3.8374e-01
  -5.0229e-02

Acl = At + F*Cy;
eig(Acl)
ans =
  -2.1170e-04
  -3.8819e-05 + 1.0470e-03i
  -3.8819e-05 - 1.0470e-03i
  -1.7360e+02

% optimal controller
ctr1 = ss(At + But(:,2) * K + F * Cyt, -F, K, 0)

a =
               x1           x2           x3           x4
   x1           0     0.001925            1            0
   x2           0   -0.0005787            0            1
   x3   4.034e-06   -1.279e-06            0      0.05157
   x4    -0.01006         0.01      -0.5253      -0.1851

b =
            u1
   x1   -577.4
   x2    173.6
   x3   0.3837
   x4  0.05023

c =
           x1      x2      x3      x4
   y1  -1.006       1  -52.52  -18.51

d =
       u1
   y1   0

Continuous-time model.

% controller transfer-function
zpk(ctr1)

Zero/pole/gain:
733.09 (s+5.507e-05) (s^2  - 0.0005306s + 9.993e-07)
---------------------------------------------------
(s+0.05429) (s+0.0006114) (s^2  + 0.1308s + 0.009978)
```

```
% compute using matlab's lqg
sys = ss(At, But(:,2), Cyt, 0);
ctr2 = lqg(sys, [Czt Dzut]' * [Czt Dzut], ...
           [Bwt; Dywt] * Wt * [Bwt; Dywt]')

a =
               x1_e        x2_e        x3_e        x4_e
   x1_e           0    0.001925           1           0
   x2_e           0  -0.0005787           0           1
   x3_e   4.034e-06  -1.279e-06           0     0.05157
   x4_e    -0.01006        0.01     -0.5253     -0.1851

b =
              y1
   x1_e   -577.4
   x2_e    173.6
   x3_e   0.3837
   x4_e  0.05023

c =
         x1_e    x2_e    x3_e    x4_e
   u1  -1.006       1  -52.52  -18.51

d =
       y1
   u1   0

Input groups:
       Name        Channels
    Measurement       1

Output groups:
      Name      Channels
    Controls       1

Continuous-time model.

% compute closed loop system
Acl = [At But(:,2)*K; -F*Cyt, At+But(:,2)*K+F*Cyt];
Bcl = [Bwt; -F*Dywt];
Ccl = [Czt; Dzut*K];

eig(Acl)
ans =
  -6.7473e-02 + 7.5883e-02i
  -6.7473e-02 - 7.5883e-02i
  -5.0086e-02
  -7.8118e-05
  -2.3280e-04 + 2.1126e-04i
  -2.3280e-04 - 2.1126e-04i
  -5.6530e-05 + 1.1651e-03i
  -5.6530e-05 - 1.1651e-03i
```

```
% state variance
Y = lyap(Acl, Bcl * Wt * Bcl')
Y =
  Columns 1 through 6
   2.4528e+08   2.4035e+01   5.3316e-07  -4.1881e+04   1.8471e+07   1.7321e+07
   2.4037e+01   2.4091e+07   4.1881e+04  -3.1618e-08   1.7321e+07   1.8883e+07
  -4.7729e-07   4.1881e+04   1.1701e+03  -9.6964e-05  -1.6668e+04   3.0368e+04
  -4.1881e+04   3.0062e-08  -9.6965e-05   1.0226e+03  -2.0345e+04  -1.5069e+03
   1.8471e+07   1.7321e+07  -1.6668e+04  -2.0345e+04   1.8471e+07   1.7321e+07
   1.7321e+07   1.8883e+07   3.0368e+04  -1.5069e+03   1.7321e+07   1.8883e+07
  -1.6668e+04   3.0368e+04   9.9675e+02   1.1612e+00  -1.6668e+04   3.0368e+04
  -2.0345e+04  -1.5069e+03   1.1612e+00   1.0205e+03  -2.0345e+04  -1.5069e+03
  Columns 7 through 8
  -1.6668e+04  -2.0345e+04
   3.0368e+04  -1.5069e+03
   9.9675e+02   1.1612e+00
   1.1612e+00   1.0205e+03
  -1.6668e+04  -2.0345e+04
   3.0368e+04  -1.5069e+03
   9.9675e+02   1.1612e+00
   1.1612e+00   1.0205e+03
sqrt(trace(Y))
ans =
   1.7514e+04
sqrt(diag(Y))
ans =
   1.5661e+04
   4.9082e+03
   3.4206e+01
   3.1977e+01
   4.2978e+03
   4.3454e+03
   3.1571e+01
   3.1945e+01

% optimal state feedback control (using u1 and u2)
rho = 1
rho =
     1
Czt = [eye(4); zeros(2,4)];
Dzut = sqrt(rho)*[zeros(4,2); eye(2)];
Czt'*Dzut
ans =
     0     0
     0     0
     0     0
     0     0
[K,X,S] = lqr(At, But, Czt'*Czt, Dzut'*Dzut);
K = - K
K =
  -9.8444e-01   1.7795e-01  -1.3843e+01  -2.4919e+00
  -1.7795e-01  -9.8404e-01  -2.4919e+00  -1.4741e+01

Acl = At + But*K;
```

```
eig(Acl)
ans =
  -7.1524e-02 + 8.2854e-02i
  -7.1524e-02 - 8.2854e-02i
  -7.1395e-02 + 5.7005e-02i
  -7.1395e-02 - 5.7005e-02i

% optimal state estimation
[F,X,S] = lqr(At', Cyt', Bwt * Wt * Bwt', Dywt * Wt * Dywt');
F = - F'
F =
   5.7737e+02
  -1.7360e+02
  -3.8374e-01
  -5.0229e-02

Acl = At + F*Cy;
eig(Acl)
ans =
  -2.1170e-04
  -3.8819e-05 + 1.0470e-03i
  -3.8819e-05 - 1.0470e-03i
  -1.7360e+02

% optimal controller
ctr1 = ss(At + But * K + F * Cyt, -F, K, 0)

a =
               x1          x2          x3          x4
   x1           0    0.001925           1           0
   x2           0  -0.0005787           0           1
   x3    -0.00984    0.001778     -0.1384     0.02665
   x4    -0.00178   -0.009841    -0.02502     -0.1474

b =
            u1
   x1   -577.4
   x2    173.6
   x3   0.3837
   x4  0.05023

c =
            x1       x2       x3       x4
   y1  -0.9844    0.178   -13.84   -2.492
   y2   -0.178   -0.984   -2.492   -14.74

d =
       u1
   y1   0
   y2   0

Continuous-time model.

% controller transfer-function
```

```
zpk(ctr1)

Zero/pole/gain from input to output...
        593.8432 (s+0.01501) (s^2  + 0.1347s + 0.009646)
 #1:  -----------------------------------------------------
      (s^2  + 0.144s + 0.008393) (s^2  + 0.1424s + 0.01202)

          -69.7822 (s+0.2538) (s+0.0984) (s-5.737e-05)
 #2:  -----------------------------------------------------
      (s^2  + 0.144s + 0.008393) (s^2  + 0.1424s + 0.01202)


% compute using matlab's lqg
sys = ss(At, But, Cyt, 0);
ctr2 = lqg(sys, [Czt Dzut]' * [Czt Dzut], ...
           [Bwt; Dywt] * Wt * [Bwt; Dywt]')

a =
               x1_e        x2_e        x3_e        x4_e
   x1_e           0    0.001925           1           0
   x2_e           0  -0.0005787           0           1
   x3_e    -0.00984    0.001778     -0.1384     0.02665
   x4_e    -0.00178   -0.009841    -0.02502     -0.1474

b =
              y1
   x1_e   -577.4
   x2_e    173.6
   x3_e   0.3837
   x4_e  0.05023

c =
          x1_e     x2_e     x3_e     x4_e
   u1  -0.9844    0.178   -13.84   -2.492
   u2   -0.178   -0.984   -2.492   -14.74

d =
       y1
   u1   0
   u2   0

Input groups:
       Name        Channels
    Measurement       1

Output groups:
      Name      Channels
    Controls      1,2

Continuous-time model.

% compute closed loop system
Acl = [At But*K; -F*Cyt, At+But*K+F*Cyt];
Bcl = [Bwt; -F*Dywt];
```

```
Ccl = [Czt; Dzut*K];

eig(Acl)
ans =
  -5.6530e-05 + 1.1651e-03i
  -5.6530e-05 - 1.1651e-03i
  -2.3280e-04 + 2.1126e-04i
  -2.3280e-04 - 2.1126e-04i
  -7.1524e-02 + 8.2854e-02i
  -7.1524e-02 - 8.2854e-02i
  -7.1395e-02 + 5.7005e-02i
  -7.1395e-02 - 5.7005e-02i

% state variance
Y = lyap(Acl, Bcl * Wt * Bcl')
Y =
  Columns 1 through 6
   2.2715e+08  -1.7437e+07   1.9480e-05  -1.7501e+04   3.4373e+05  -1.1556e+05
  -1.7437e+07   5.2476e+06   1.7501e+04   1.3480e-07  -1.1556e+05   3.9555e+04
  -1.9472e-05   1.7501e+04   1.3688e+03  -3.4619e+02  -1.6668e+04   5.9891e+03
  -1.7501e+04  -1.3478e-07  -3.4619e+02   1.1253e+02   4.0341e+03  -1.5069e+03
   3.4373e+05  -1.1556e+05  -1.6668e+04   4.0341e+03   3.4373e+05  -1.1556e+05
  -1.1556e+05   3.9555e+04   5.9891e+03  -1.5069e+03  -1.1556e+05   3.9555e+04
  -1.6668e+04   5.9891e+03   1.1955e+03  -3.4503e+02  -1.6668e+04   5.9891e+03
   4.0341e+03  -1.5069e+03  -3.4503e+02   1.1046e+02   4.0341e+03  -1.5069e+03
  Columns 7 through 8
  -1.6668e+04   4.0341e+03
   5.9891e+03  -1.5069e+03
   1.1955e+03  -3.4503e+02
  -3.4503e+02   1.1046e+02
  -1.6668e+04   4.0341e+03
   5.9891e+03  -1.5069e+03
   1.1955e+03  -3.4503e+02
  -3.4503e+02   1.1046e+02
sqrt(trace(Y))
ans =
   1.5257e+04
sqrt(diag(Y))
ans =
   1.5071e+04
   2.2908e+03
   3.6997e+01
   1.0608e+01
   5.8629e+02
   1.9889e+02
   3.4576e+01
   1.0510e+01

diary off
```