3 LQR with Noisy Input

Problem:
Compute a state feedback controller

$$u(t) = Kx(t)$$

that stabilizes the closed loop system and minimizes

$$J := \lim_{t\to\infty} E\left[x(t)^T Q x(t) + u(t)^T R u(t)\right]$$

for the LTI system

$$\dot{x}(t) = Ax(t) + B_u u(t) + B_w w(t), \quad x(0) = 0$$

Assumptions:

a) $Q \succeq 0$, $R \succ 0$;

b) (A, B_u) stabilizable;

c) $w(t)$ is a Gaussian white noise vector with zero mean and covariance $W \succ 0$.

A first step toward a solution:
The closed loop cost is

$$J = \lim_{t\to\infty} E\left[x(t)^T (Q + K^T R K) x(t)\right]$$

and the closed system is

$$\dot{x} = (A + B_u K)x + B_w w, \quad x(0) = 0$$

As before, we seek to relate J to a Gramian.

3.1 System Response to a Noisy Input

Problem: Evaluate

$$J := \lim_{t\to\infty} E\left[x(t)^T Q x(t)\right]$$

for the LTI system

$$\dot{x}(t) = Ax(t) + B_w w(t), \quad x(0) = 0$$

where w is a Gaussian white noise vector with zero mean and covariance matrix $W \succ 0$, i.e.,

$$E\left[w(t)\right] = 0, \qquad E\left[w(t)w(\tau)^T\right] = W\delta(t-\tau).$$

Solution:
Recall that trace is linear and that $\mathrm{trace}(AB) = \mathrm{trace}(BA)$ so that

$$x^T x = \mathrm{trace}(x^T x) = \mathrm{trace}(x x^T)$$

and

$$\begin{aligned} J &= \lim_{t\to\infty} E\left[x(t)^T Q x(t)\right], \\ &= \lim_{t\to\infty} E\left[\mathrm{trace}\left(Q x(t) x(t)^T\right)\right]. \end{aligned}$$

Furthermore

$$x(t) = \int_0^t e^{A(t-\tau)} B_w w(\tau)\, d\tau$$

so that

$$\begin{aligned} J &= \lim_{t\to\infty} E\left[\mathrm{trace}\left(Q \int_0^t e^{A(t-\tau)} B_w w(\tau)\, d\tau \int_0^t w^T(\sigma) B_w^T e^{A^T(t-\sigma)}\, d\sigma\right)\right], \\ &= \lim_{t\to\infty} \mathrm{trace}\left(Q \int_0^t e^{A(t-\tau)} B_w \int_0^t E\left[w(\tau)w^T(\sigma)\right] B_w^T e^{A^T(t-\sigma)}\, d\sigma\, d\tau\right), \\ &= \lim_{t\to\infty} \mathrm{trace}\left(Q \int_0^t e^{A(t-\tau)} B_w \int_0^t W\delta(\tau-\sigma) B_w^T e^{A^T(t-\sigma)}\, d\sigma\, d\tau\right), \end{aligned}$$

because $w(t)$ is white.

Integrating on σ

$$\begin{aligned}
J &= \lim_{t\to\infty} \operatorname{trace}\left(Q \int_0^t e^{A(t-\tau)} B_w W B_w^T e^{A^T(t-\tau)} \, d\tau \right), \\
&= \lim_{t\to\infty} \operatorname{trace}\left(Q \int_0^t e^{A\eta} B_w W B_w^T e^{A^T \eta} \, d\eta \right), \qquad \eta := t - \tau \\
&= \lim_{t\to\infty} \operatorname{trace}\left(\left[\int_0^t e^{A^T\eta} Q e^{A\eta} \, d\eta \right] B_w W B_w^T \right), \\
&= \operatorname{trace}\left(\left[\int_0^\infty e^{A^T\eta} Q e^{A\eta} \, d\eta \right] B_w W B_w^T \right), \\
&= \operatorname{trace}\left(X B_w W B_w^T \right),
\end{aligned}$$

where X is the solution to the Lyapunov equation

$$A^T X + XA + Q = 0.$$

Back to our problem, the closed loop cost is

$$J = \lim_{t\to\infty} E\left[x(t)^T(Q + K^T RK)x(t)\right]$$

and the closed system is

$$\dot{x} = (A + B_u K)x + B_w w, \quad x(0) = 0$$

Therefore J can be computed as

$$J = \operatorname{trace}\left(X B_w W B_w^T\right)$$

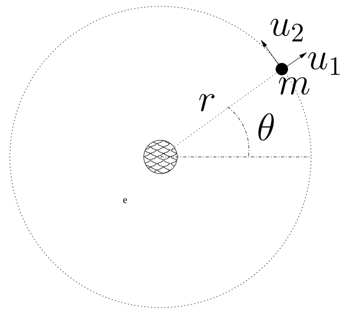
where

$$(A + B_u K)^T X + X(A + B_u K) + Q + K^T RK = 0.$$

As before, the optimal control is again computed as

$$A^T X + XA - XB_u R^{-1} B_u^T X + Q = 0, \qquad K = -R^{-1} B_u^T X.$$

3.2 Example: controlling a satellite in circular orbit



Satellite of mass m with thrust in the radial direction u_1 and in the tangential direction u_2. Continuing...

$$m(\ddot{r} - r\dot{\theta}^2) = u_1 - \frac{km}{r^2} + w_1,$$
$$m(2\dot{r}\dot{\theta} + r\ddot{\theta}) = u_2 + w_2,$$

where w_1 and w_2 are independent white noise disturbances with variances δ_1 and δ_2.
As before, putting in state space and linearizing around

$$\bar{x}(t) = \begin{pmatrix} \bar{r} \\ \bar{\omega} t \\ 0 \\ \bar{\omega} \end{pmatrix}, \qquad \bar{u} = \begin{pmatrix} \bar{u}_1 \\ \bar{u}_2 \end{pmatrix} = 0, \qquad \bar{w} = \begin{pmatrix} \bar{w}_1 \\ \bar{w}_2 \end{pmatrix} = 0, \qquad \bar{\omega} = \sqrt{k/\bar{r}^3}$$

one obtains the linearized system

$$\dot{x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 3\,\bar{\omega}^2 & 0 & 0 & 2\,\bar{r}\bar{\omega} \\ 0 & 0 & -2\,\bar{\omega}/\bar{r} & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1/m & 0 \\ 0 & 1/(m\bar{r}) \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1/m & 0 \\ 0 & 1/(m\bar{r}) \end{bmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.$$

Problem: Given

$$m = 100 \text{ kg}, \qquad \bar{r} = R + 300 \text{ km}, \qquad \bar{k} = GM$$

where $G \approx 6.673 \times 10^{-11}$ N m^2/kg^2 is the universal gravitational constant, and $M \approx 5.98 \times 10^{24}$ kg and $R \approx 6.37 \times 10^3$ km are the mass and radius of the earth. If the variances δ_1 and δ_2 are $0.1N$ find solutions to the LQR control problem where

$$Q = I, \qquad R = \rho I,$$

using u_2 only first, then using u_1 and u_2.

```
% MAE 280 B - Linear Control Design
% Mauricio de Oliveira
% LQR Control - Part I
m = 100       % 100 kg
m =
   100
r = 300E3     % 300 km
r =
      300000
R = 6.37E6    % 6.37 10^3 km
R =
     6370000
G = 6.673E-11 % 6.673 N m^2/kg^2
G =
   6.6730e-11
M = 5.98E24   % 5.98 10^24 kg
M =
   5.9800e+24

% gravitational force constant
k = G * M
k =
   3.9905e+14

% angular velocity (rad/s)
w = sqrt(k/((R+r)^3))
w =
   1.1596e-03

% "ground" velocity (m/s)
v = w * (R + r)
v =
   7.7348e+03

% linearized system matrices
A = [0 0 1 0; 0 0 0 1; 3*w^2 0 0 2*(r+R)*w; 0 0 -2*w/(r+R) 0]
A =
            0            0   1.0000e+00            0
            0            0            0   1.0000e+00
   4.0343e-06            0            0   1.5470e+04
            0            0  -3.4772e-10            0
Bu = [0 0; 0 0; 1/m 0; 0 1/(m*r)]
Bu =
            0            0
            0            0
   1.0000e-02            0
            0   3.3333e-08
Bw = [0 0; 0 0; 1/m 0; 0 1/(m*r)]
Bw =
            0            0
            0            0
   1.0000e-02            0
            0   3.3333e-08
```

```
% noise variances
W = 0.1 * eye(2)
W =
   1.0000e-01            0
            0   1.0000e-01

% open loop system eigenvalues
eig(A)
ans =
            0
            0
            0 + 1.1596e-03i
            0 - 1.1596e-03i

% LQR weights
Q = diag([1 1 1 1])
Q =
     1     0     0     0
     0     1     0     0
     0     0     1     0
     0     0     0     1
R = eye(1)
R =
     1

% Cheapest control (using u2 only)
rho = 1e6
rho =
     1000000
[K,X,S] = lqr(A, Bu(:,2), Q, rho * R);
K = - K
K =
  -1.1246e-03   1.0000e-03  -2.4595e-01  -4.7779e+05
trace(X * Bw * W * Bw')
ans =
   1.6500e+03

% closed loop eigenvalues
Acl = A + Bu(:,2)*K;
eig(Acl)
ans =
  -7.9630e-03
  -3.9816e-03 + 6.9930e-03i
  -3.9816e-03 - 6.9930e-03i
  -2.6079e-10

% Cheapest control (using u1 only)
% WARNING: Recall this can't be done because
% (A,Bu(:,1) is not controllable
% (in this case also not stabilizable!)
rho = 1
rho =
     1
```

```
[K,X,S] = lqr(A, Bu(:,1), Q, rho * R);
K = - K
K =
  -7.6736e-01   6.4170e-01  -1.4176e+01   6.6866e+08
trace(X * Bw * W * Bw')
ans =
   5.1528e+11

% closed loop eigenvalues
Acl = A + Bu(:,1)*K;
eig(Acl)
ans =
  -7.0882e-02 + 7.0538e-02i
  -7.0882e-02 - 7.0538e-02i
   8.1403e-18
  -2.2313e-10

diary off
```

3.3 More on System Response to a Noisy Input

Problem: Evaluate the state mean and asymptotic covariance matrix

$$Y := \lim_{t\to\infty} E\left[x(t)x(t)^T\right]$$

for the LTI system

$$\dot{x}(t) = Ax(t) + B_w w(t), \quad x(0) = 0$$

where w is a Gaussian white noise vector with zero mean and covariance matrix $W \succ 0$, i.e.,

$$E\left[w(t)\right] = 0, \qquad E\left[w(t)w(\tau)^T\right] = W\delta(t-\tau).$$

Solution:
As before

$$x(t) = \int_0^t e^{A(t-\tau)} B_w w(\tau)\, d\tau$$

For the mean

$$E\left[x(t)\right] = E\left[\int_0^t e^{A(t-\tau)} B_w w(\tau)\, d\tau\right] = \int_0^t e^{A(t-\tau)} B_w E\left[w(\tau)\right]\, d\tau = 0$$

For the variance

$$\begin{aligned}
Y &= \lim_{t\to\infty} E\left[\int_0^t e^{A(t-\tau)} B_w w(\tau)\, d\tau \int_0^t w^T(\sigma) B_w^T e^{A^T(t-\sigma)}\, d\sigma\right], \\
&= \lim_{t\to\infty} \int_0^t e^{A(t-\tau)} B_w \int_0^t E\left[w(\tau)w^T(\sigma)\right] B_w^T e^{A^T(t-\sigma)}\, d\sigma\, d\tau, \\
&= \lim_{t\to\infty} \int_0^t e^{A(t-\tau)} B_w \int_0^t W\delta(\tau-\sigma) B_w^T e^{A^T(t-\sigma)}\, d\sigma\, d\tau, \\
&= \lim_{t\to\infty} \int_0^t e^{A(t-\tau)} B_w W B_w^T e^{A^T(t-\tau)}\, d\tau, \\
&= \lim_{t\to\infty} \int_0^t e^{A\eta} B_w W B_w^T e^{A^T\eta}\, d\eta, \qquad \eta := t-\tau
\end{aligned}$$

where Y is the solution to the Lyapunov equation

$$AY + YA^T + B_w W B_w^T = 0.$$

```
% MAE 280 B - Linear Control Design
% Mauricio de Oliveira
%
% LQR Control - Part II
%
m = 100;                % 100 kg
r = 300E3;              % 300 km
R = 6.37E6;             % 6.37 10^3 km
G = 6.673E-11;          % 6.673 N m^2/kg^2
M = 5.98E24;            % 5.98 10^24 kg
k = G * M;              % gravitational force constant
w = sqrt(k/((R+r)^3)); % angular velocity (rad/s)
v = w * (R + r);        % "ground" velocity (m/s)

% linearized system matrices
A = [0 0 1 0; 0 0 0 1; 3*w^2 0 0 2*(r+R)*w; 0 0 -2*w/(r+R) 0];
Bu = [0 0; 0 0; 1/m 0; 0 1/(m*r)];
Bw = [0 0; 0 0; 1/m 0; 0 1/(m*r)];

% noise variances
W = 0.1 * eye(2)
W =
   1.0000e-01            0
            0   1.0000e-01

% scale
T = diag([1 r 1 r])
T =
           1           0           0           0
           0      300000           0           0
           0           0           1           0
           0           0           0      300000

% similarity transformation
At = T * A / T
At =
            0            0   1.0000e+00            0
            0            0            0   1.0000e+00
   4.0343e-06            0            0   5.1565e-02
            0            0  -1.0432e-04            0
But = T * Bu
But =
            0            0
            0            0
   1.0000e-02            0
            0   1.0000e-02
Bwt = T * Bw
Bwt =
            0            0
            0            0
   1.0000e-02            0
            0   1.0000e-02
```

```
% LQR weights
Q = diag([1 1 1 1])
Q =
     1     0     0     0
     0     1     0     0
     0     0     1     0
     0     0     0     1
R = eye(1)
R =
     1

% Cheapest control (using u2 only)
rho = 1e6
rho =
     1000000
[Kt,X,S] = lqr(At, But(:,2), Q, rho * R);
Kt = - Kt
Kt =
  -1.1466e-03   1.0000e-03  -2.6902e-01  -1.6045e+00
trace(X * Bwt * W * Bwt')
ans =
   1.7243e+03

% closed look eigenvalues
Acl = At + But(:,2)*Kt;
eig(Acl)
ans =
  -4.0160e-03 + 6.9969e-03i
  -4.0160e-03 - 6.9969e-03i
  -7.9348e-03
  -7.8118e-05

% gain in original coordinates
K = Kt * T
K =
  -1.1466e-03   3.0000e+02  -2.6902e-01  -4.8135e+05

% state covariance
Y = lyap(Acl, Bwt * W * Bwt')
Y =
   3.1751e+02   2.4035e+01   1.3323e-15  -2.1795e-01
   2.4035e+01   4.0038e+01   2.1795e-01  -1.5613e-17
   1.3323e-15   2.1795e-01   9.9579e-03  -9.6965e-05
  -2.1795e-01  -1.5613e-17  -9.6965e-05   4.8426e-04
sqrt(trace(Y))
ans =
   1.8909e+01
sqrt(diag(Y))
ans =
   1.7819e+01
   6.3276e+00
   9.9789e-02
   2.2006e-02
```

```
% optimal control (using u2 only)
rho = 1
rho =
     1
[Kt,X,S] = lqr(At, But(:,2), Q, rho * R);
Kt = - Kt
Kt =
  -1.0055e+00   1.0000e+00  -5.2522e+01  -1.8511e+01
trace(X * Bwt * W * Bwt')
ans =
   4.8525e+01

% closed loop eigenvalues
Acl = At + But(:,2)*Kt;
eig(Acl)
ans =
  -6.7473e-02 + 7.5883e-02i
  -6.7473e-02 - 7.5883e-02i
  -5.0086e-02
  -7.8118e-05

% gain in original coordinates
K = Kt * T
K =
  -1.0055e+00   3.0000e+05  -5.2522e+01  -5.5533e+06

% state covariance
Y = lyap(Acl, Bwt * W * Bwt')
Y =
   2.4002e+01   2.4035e+01  -8.4655e-16  -5.9055e-03
   2.4035e+01   2.4416e+01   5.9055e-03  -1.8874e-15
  -8.4655e-16   5.9055e-03   2.0769e-04  -9.6965e-05
  -5.9055e-03  -1.8874e-15  -9.6965e-05   6.2298e-04
sqrt(trace(Y))
ans =
   6.9584e+00
sqrt(diag(Y))
ans =
   4.8991e+00
   4.9413e+00
   1.4412e-02
   2.4960e-02

% optimal control (using u1 and u2)
R = eye(2);
rho = 1
rho =
     1
[Kt,X,S] = lqr(At, But, Q, rho * R);
Kt = - Kt
Kt =
  -9.8444e-01   1.7795e-01  -1.3843e+01  -2.4919e+00
  -1.7795e-01  -9.8404e-01  -2.4919e+00  -1.4741e+01
trace(X * Bwt * W * Bwt')
```

```
ans =
   2.8584e-02

% closed loop eigenvalues
Acl = At + But*Kt;
eig(Acl)
ans =
  -7.1524e-02 + 8.2854e-02i
  -7.1524e-02 - 8.2854e-02i
  -7.1395e-02 + 5.7005e-02i
  -7.1395e-02 - 5.7005e-02i

% gain in original coordinates
K = Kt * T
K =
  -9.8444e-01   5.3386e+04  -1.3843e+01  -7.4756e+05
  -1.7795e-01  -2.9521e+05  -2.4919e+00  -4.4223e+06

% state covariance
Y = lyap(Acl, Bwt * W * Bwt')
Y =
   3.6095e-03   9.3241e-18            0  -4.3573e-05
   9.3241e-18   3.3896e-03   4.3573e-05  -3.9980e-19
            0   4.3573e-05   3.6680e-05   4.0658e-20
  -4.3573e-05  -3.9980e-19   4.0658e-20   3.4445e-05
sqrt(trace(Y))
ans =
   8.4084e-02
sqrt(diag(Y))
ans =
   6.0079e-02
   5.8220e-02
   6.0564e-03
   5.8690e-03

diary off
```