2 The Linear Quadratic Regulator (LQR)

Problem:
Compute a state feedback controller

$$u(t) = Kx(t)$$

that stabilizes the closed loop system and minimizes

$$J := \int_0^\infty x(t)^T Q x(t) + u(t)^T R u(t)\, dt$$

where x and u are the state and control of the LTI system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0.$$

Assumptions:

a) $Q \succeq 0$, $R \succ 0$;

b) (A, B) stabilizable;

A first step toward a solution:
The closed loop cost is

$$J = \int_0^\infty x(t)^T (Q + K^T R K) x(t)\, dt$$

and the closed loop system is

$$\dot{x} = (A + BK)x, \quad x(0) = x_0.$$

But for a given K and x_0

$$x(t) = e^{(A+BK)t} x_0.$$

Hence

$$\begin{aligned} J &= \int_0^\infty x_0^T e^{(A+BK)^T t} (Q + K^T R K) e^{(A+BK)t} x_0\, dt \\ &= x_0^T \left(\int_0^\infty e^{(A+BK)^T t} (Q + K^T R K) e^{(A+BK)t}\, dt \right) x_0. \end{aligned}$$

This means that J can be computed as

$$J = x_0^T X x_0$$

where X is the solution to the Lyapunov equation

$$(A + BK)^T X + X(A + BK) + Q + K^T RK = 0.$$

Before proceeding we need to learn how to solve the above Lyapunov equation in X *and* K. This is not always possible. In this case, because $R \succ 0$, we can complete the squares, rewriting the above equation in the form

$$A^T X + XA - XBR^{-1}B^T X + Q + (XBR^{-1} + K^T)R(R^{-1}B^T X + K) = 0.$$

Note that K is confined to the term

$$(XBR^{-1} + K^T)R(R^{-1}B^T X + K) \succeq 0$$

and that for

$$K = -R^{-1}B^T X.$$

we have

$$Q + (XBR^{-1} + K^T)R(R^{-1}B^T X + K) = Q.$$

This reduces the above equation to

$$A^T X + XA - XBR^{-1}B^T X + Q = 0.$$

This is an Algebraic Riccati Equation (ARE) in X.

As we learn more about AREs we shall prove that the above choice of K and X is so that

a) $A + BK$ is Hurwitz (asymptotically stable);

b) X is "minimum" in a certain sense;

c) The associated J is minimized.

2.1 Comparison Lemma

If $S \succeq 0$ and

$$Q_2 \succeq Q_1 \succeq 0$$

then X_1 and X_2, solutions to the Riccati equations

$$\begin{aligned} A^T X_1 + X_1 A - X_1 S X_1 + Q_1 = 0, \\ A^T X_2 + X_2 A - X_2 S X_2 + Q_2 = 0, \end{aligned}$$

are such that

$$X_2 \succeq X_1$$

if $A - SX_2$ is asymptotically stable.

Proof: Note that

$$\begin{aligned} & A^T X_1 + X_1 A - X_1 S X_1 + Q_1 \\ & \qquad = (A - SX_2)^T X_1 + X_1(A - SX_2) + X_2 S X_2 + Q_1 - (X_1 - X_2) S (X_1 - X_2), \end{aligned}$$

and

$$A^T X_2 + X_2 A - X_2 S X_2 + Q_2 = (A - SX_2)^T X_2 + X_2(A - SX_2) + X_2 S X_2 + Q_2$$

Now subtract the above equations to obtain the Lyapunov equation

$$(A - SX_2)^T \bar{X} + \bar{X}(A - SX_2) + \bar{Q} = 0$$

where

$$\bar{X} := X_2 - X_1, \qquad \bar{Q} := (Q_2 - Q_1) + (X_1 - X_2) S (X_1 - X_2) \succeq 0.$$

Therefore, if $A - SX_2$ is Hurwitz we conclude that $\bar{X} = X_2 - X_1 \succeq 0$, that is $X_2 \succeq X_1$.

We can now use the comparison lemma to compare the two AREs

$$A^T X_2 + X_2 A - X_2 B R^{-1} B^T X_2 + Q_2 = 0$$

and

$$A^T X_1 + X_1 A - X_1 B R^{-1} B^T X_1 + Q_1 = 0$$

where

$$S = B R^{-1} B^T \succeq 0,$$

and

$$Q_1 = Q, \qquad Q_2 = Q + (X_2 B R^{-1} + K^T) R (R^{-1} B^T X_2 + K).$$

Note that for any X_2 and stabilizing K that

$$Q_2 = Q + (X_2 B R^{-1} + K^T) R (R^{-1} B^T X_2 + K) \succeq Q = Q_1$$

because $R \succ 0$. Therefore, for any choice of

$$K \neq -R^{-1} B^T X_1$$

we shall have

$$X_2 \succeq X_1.$$

This proves that X_1 is "minimum". Of course this also implies that

$$J_2 = x_0^T X_2 x_0 \geq x_0^T X_1 x_0 = J_1$$

so that J is also being minimized.

2.2 More on AREs

Warning: In this section we consider Riccati equations of the form

$$A^T X + XA + XZX + Q = 0$$

Lemma 1: Consider the *Hamiltonian matrix*

$$H := \begin{bmatrix} A & Z \\ -Q & -A^T \end{bmatrix}.$$

where A, $Z = Z^T$ and $Q = Q^T \in \mathbb{R}^{n\times n}$.

1. λ is an eigenvalue of H if and only if $-\lambda$ is an eigenvalue of H.
2. If H has no eigenvalues on the imaginary axis then there exists a matrix $W \in \mathbb{R}^{n\times n}$ such that

$$HV_1 = V_1 W \tag{2}$$

where W is Hurwitz.

Proof:
Item 1. H has eigenvalues pairs which are symmetric w.r.t the imaginary axis because

$$J^{-1}HJ = -JHJ = -H^T, \qquad J := \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}, \quad J^{-1} = -J$$

Item 2. Let H_J be the Jordan form of matrix H so that

$$HV = VH_J$$

where $V \in \mathbb{R}^{2n\times 2n}$ is a matrix whose columns are the (generalized) eigenvectors of H. Since the eigenvalues of H are symmetric with respect to the imaginary axis and there are no eigenvalues on the imaginary axis, there exists at least two distinct Jordan blocks

$$H \begin{bmatrix} V_1 & V_2 \end{bmatrix} = \begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} H_{J_-} & 0 \\ 0 & H_{J_+} \end{bmatrix}$$

where all n eigenvalues of H_{J_-} have negative real part, i.e., H_{J_-} is Hurwitz. The first columns of the above equation are in the form (2) with $W = H_{J_-}$ Hurwitz.

Lemma 2: Consider the Algebraic Riccati Equation (ARE)

$$A^T X + XA + XZX + Q = 0$$

where A, $Z = Z^T$ and $Q = Q^T \in \mathbb{R}^{n \times n}$ and the associated *Hamiltonian matrix*

$$H := \begin{bmatrix} A & Z \\ -Q & -A^T \end{bmatrix}.$$

which is assumed to have no eigenvalue on the imaginary axis.

1. Let
$$V_1 = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \in \mathbb{C}^{2n \times n}$$
be (generalized) eigenvectors of H associated with all n eigenvalues with negative real part. If X_1 is nonsingular then $X = X_2 X_1^{-1}$ solves the ARE.

2. The solution obtained in item 1. is
   (a) real,
   (b) symmetric,
   (c) unique stabilizing ($A + ZX$ is Hurwitz).

3. If $Z \succeq 0$ (or $Z \preceq 0$) then X_1 is invertible if and only if (A, Z) is stabilizable.

Proof:
Item 1. From Item 2. of Lemma 1 there exists a Hurwitz matrix W such that

$$HV_1 = V_1 W$$

Then, multiplying the above by X_1^{-1} on the right and by $\begin{bmatrix} X & -I \end{bmatrix}$ on the left we get

$$\begin{bmatrix} X & -I \end{bmatrix} H \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} X & -I \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix} X_1 W X_1^{-1} = 0$$

Note that

$$\begin{bmatrix} X & -I \end{bmatrix} H \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} X & -I \end{bmatrix} \begin{bmatrix} A & Z \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix} = A^T X + XA + XZX + Q$$

Item 2. (a) Since H is real, the columns of V_1 can be chosen complex conjugates in pairs so that

$$\bar{V}_1 = \begin{bmatrix} \bar{X}_1 \\ \bar{X}_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} P = \begin{bmatrix} X_1 P \\ X_2 P \end{bmatrix}$$

where P is some permutation matrix and

$$\bar{X} = \bar{X}_2 \bar{X}_1^{-1} = X_2 P P^{-1} X_1^{-1} = X_2 X_1^{-1} = X$$

that is X is real.

Item 2. (b) X is symmetric if

$$X = X_2 X_1^{-1} = (X_1^{-1})^* X_2^* = X^*$$

or in other words

$$T = X_2^* X_1 - X_1^* X_2 = 0.$$

Now note that

$$T = V_1^* J V_1 = \begin{bmatrix} X_1^* & X_2^* \end{bmatrix} \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

Since $JHJ = H^T$, H and J satisfy the Lyapunov equation

$$JH + H^T J = 0$$

Multiplying the above equation by V_1^* on the left and V_1 on the right and using (2)

$$\begin{aligned} 0 &= V_1^* J H V_1 + V_1^* H^T J V_1 \\ &= V_1^* J V_1 W + W^* V_1^* J V_1 \\ &= TW + W^* T. \end{aligned}$$

Because W is Hurwitz $T = 0$.

Item 2. (c) Multiply (2) by X_1^{-1} on the right and by $\begin{bmatrix} I & 0 \end{bmatrix}$ on the left to obtain

$$\begin{bmatrix} I & 0 \end{bmatrix} H \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} A & Z \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix} = A + ZX = X_1 W X_1^{-1}.$$

Therefore, $A + ZX$ is stable because it is similar to a stable matrix. For uniqueness assume $\tilde{X}$ is also a stabilizing solution to the ARE. Therefore subtracting the two AREs

$$\begin{aligned} 0 &= (A^T X + XA + XZX + Q) - (A^T \tilde{X} + \tilde{X} A + \tilde{X} Z \tilde{X} + Q) \\ &= A^T (X - \tilde{X}) + (X - \tilde{X}) A + XZX - \tilde{X} Z \tilde{X} \\ &= A^T (X - \tilde{X}) + (X - \tilde{X}) A + XZX - \tilde{X} Z \tilde{X} - XZ\tilde{X} + XZ\tilde{X} \\ &= A^T (X - \tilde{X}) + (X - \tilde{X}) A + XZ(X - \tilde{X}) + (X - \tilde{X}) Z \tilde{X} \\ &= (A + ZX)^T (X - \tilde{X}) + (X - \tilde{X})(A + Z\tilde{X}) \end{aligned}$$

The last equation can be seen as a Sylvester equation in $(X - \tilde{X})$ and since $A + ZX$ and $A + Z\tilde{X}$ are both Hurwitz $\lambda_i(A + ZX) + \lambda_j(A + Z\tilde{X}) < 0$ so that it admits only the trivial solution, that is, $X - \tilde{X} = 0$.

Item 3. To prove sufficiency note that if X_1 is invertible then $A + ZX$ is stable such that (A, Z) is stabilizable.
The proof of necessity is more complicated. Assume that $Z \succeq 0$ (or $Z \preceq 0$), (A, Z) is stabilizable and that X_1 is singular, such that there exists $x \neq 0$ such that $X_1 x = 0$. Multiply (2) by $\begin{bmatrix} I & 0 \end{bmatrix}$ on the left to obtain

$$\begin{bmatrix} I & 0 \end{bmatrix} H \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = AX_1 + ZX_2 = X_1 W$$

Multiply on the left by $x^* X_2^*$ and on the right by x

$$x^* X_2^* A X_1 x + x^* X_2^* Z X_2 x = x^* X_2^* X_1 W x = x^* X_1^* X_2 W x$$

and use the fact that $X_1 x = 0$ to obtain

$$x^* X_2^* Z X_2 x = 0$$

which implies $ZX_2 x = 0$ because $Z \succeq 0$ (or $Z \preceq 0$). Note that this also implies

$$X_1 W x = 0.$$

Auxiliary lemma: Assume W is Hurwitz. There exists $x \neq 0$ such that $X_1 x = X_1 W x = 0$ if and only if there exists $\tilde{x} \neq 0$ such that

$$X_1 \tilde{x} = 0, \quad W\tilde{x} = \tilde{\lambda}\tilde{x}, \quad \tilde{\lambda} + \tilde{\lambda}^* < 0.$$

Proof (Auxiliary lemma): Sufficiency is immediate since

$$X_1 W \tilde{x} = \tilde{\lambda} X_1 \tilde{x} = 0.$$

Necessity follows by contradiction. If X_1 is singular and

$$\nexists \tilde{x} : X_1 \tilde{x} = 0, \quad W\tilde{x} = \tilde{\lambda}\tilde{x}, \quad \tilde{\lambda} + \tilde{\lambda}^* < 0$$

then

$$X_1 W \tilde{x} = \tilde{\lambda} X_1 \tilde{x} \neq 0$$

for any eigenvalue/eigenvector pair $(\tilde{\lambda}, \tilde{x})$. Because W is nonsingular this must be true for n linearly independent vectors, which implies that X_1 is not singular.

□ (Auxiliary lemma)

Now multiply (2) by $\begin{bmatrix} 0 & I \end{bmatrix}$ on the left to obtain

$$\begin{bmatrix} 0 & I \end{bmatrix} H \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = -QX_1 - A^* X_2 = X_2 W$$

and multiply by $\tilde{x}$ such that $X_1 \tilde{x} = 0$ on the right

$$\begin{aligned} 0 &= (A^* X_2 + X_2 W)\tilde{x} \\ &= (A^* - \lambda I) X_2 \tilde{x}, \qquad \lambda = -\tilde{\lambda}, \qquad \lambda + \lambda^* > 0. \end{aligned}$$

Since $Z X_2 \tilde{x} = 0$, this implies that (A, Z) is not stabilizable, which is a contradiction.

We now use Lemma 2 to prove that $A + BK$ is Hurwitz. It amounts to apply item 3 of Lemma 2 to the ARE

$$A^T X + XA - XBR^{-1}B^T X + Q = 0.$$

For that notice that

$$Z = -BR^{-1}B^T \preceq 0$$

because $R \succ 0$ then $BR^{-1}B^T \succeq 0$. Therefore, if $(A, Z) = (A, -BR^{-1}B^T)$ is stabilizable then X_1 in Lemma 2 should be invertible and the solution X should be unique, symmetric and stabilizing.
With that in mind suppose that (A, B) is stabilizable but that $(A, -BR^{-1}B^T)$ is not, so that there exists $z \neq 0$ such that

$$z^* A = \lambda z^*, \qquad z^* BR^{-1}B^T = 0, \qquad \lambda + \lambda^* \geq 0.$$

Therefore

$$z^* BR^{-1}B^T z = 0.$$

Because $R^{-1} \succ 0$ this can only be true if $z^* B = 0$, which contradicts the hypothesis that (A, B) is stabilizable, proving that $(A, -BR^{-1}B^T)$ is stabilizable and

$$A + BK = A + B(-R^{-1}B^T X) = A - BR^{-1}B^T X = A + ZX$$

is Hurwitz.

Warning: The optimal control gain K is independent from the initial condition! However, the optimal cost is not!