

## 11 Output Feedback Control

Problem: Compute an output feedback controller

$$\begin{aligned}\dot{x}_c &= A_c x_c + B_c y, & x_c(0) &= 0, & x_c &\in \mathbb{R}^{n_c} \\ u &= C_c x_c + D_c y,\end{aligned}$$

that stabilizes the closed loop system and minimizes

$$J := \lim_{t \rightarrow \infty} E [z(t)^T z(t)]$$

for the LTI system

$$\begin{aligned}\dot{x} &= Ax + B_w w + B_u u, & x(0) &= 0, & x &\in \mathbb{R}^n \\ z &= C_z x + D_{zw} w + D_{zu} u, \\ y &= C_y x + D_{yw} w,\end{aligned}$$

Assumptions:

1.  $(A, B_u)$  stabilizable,
2.  $(A, C_y)$  detectable,
3.  $D_{zu}^T D_{zu} \succ 0$ ,
4.  $D_{yw} W D_{yw}^T \succ 0$ ,
5.  $w(t)$  is a Gaussian zero mean white noise with variance  $W \succ 0$ .

## 11.1 The Closed Loop System

The closed loop system is

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{x}_c \end{pmatrix} &= \begin{bmatrix} A + B_u D_c C_y & B_u C_c \\ B_c C_y & A_c \end{bmatrix} \begin{pmatrix} x \\ x_c \end{pmatrix} + \begin{bmatrix} B_w + B_u D_c D_{yw} \\ B_c D_{yw} \end{bmatrix} w, \\ z &= [C_z + D_{zu} D_c C_y \quad D_{zu} C_c] \begin{pmatrix} x \\ x_c \end{pmatrix} + [D_{zw} + D_{zu} D_c D_{yw}] w \end{aligned} \quad (19)$$

For ease of notation define

$$\tilde{x} := \begin{pmatrix} x \\ x_c \end{pmatrix}$$

so that the closed loop is of the form

$$\begin{pmatrix} \dot{\tilde{x}} \\ z \end{pmatrix} = \left[ \begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & \tilde{D} \end{array} \right] \begin{pmatrix} \tilde{x} \\ w \end{pmatrix}$$

Note that

$$\begin{aligned} \left[ \begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & \tilde{D} \end{array} \right] &= \left[ \begin{array}{cc|c} A & 0 & B_w \\ 0 & 0 & 0 \\ \hline C_z & 0 & D_{zw} \end{array} \right] + \left[ \begin{array}{cc|c} B_u & 0 \\ 0 & I \\ \hline D_{zu} & 0 \end{array} \right] \begin{bmatrix} D_c & C_c \\ B_c & A_c \end{bmatrix} \left[ \begin{array}{cc|c} C_y & 0 & D_{yw} \\ 0 & I & 0 \end{array} \right], \\ &= \left[ \begin{array}{c|c} \mathcal{A} & \mathcal{B}_w \\ \hline \mathcal{C}_z & \mathcal{D}_{zw} \end{array} \right] + \left[ \begin{array}{c|c} \mathcal{B}_u \\ \hline \mathcal{D}_{zu} \end{array} \right] \mathcal{K} \quad [ \mathcal{C}_y \mid \mathcal{D}_{yw} ]. \end{aligned}$$

The conclusion is that a closed loop system under dynamic output feedback can be recast as the connection of the augmented system

$$\begin{aligned} \dot{\tilde{x}} &= \mathcal{A} \tilde{x} + \mathcal{B}_u \tilde{u} + \mathcal{B}_w w \\ \tilde{y} &= \mathcal{C}_y \tilde{x} + \mathcal{D}_{yw} w \\ z &= \mathcal{C}_z \tilde{x} + \mathcal{D}_{zu} \tilde{u}, \end{aligned}$$

in feedback with the *static* output feedback controller

$$\tilde{u} = \mathcal{K} \tilde{y}.$$

Unfortunately we do not know much about output feedback problems :(.

## 11.2 The Cost Function

Now write the cost function

$$J = \lim_{t \rightarrow \infty} E [z(t)^T z(t)].$$

in terms of the closed loop system (19). Computing  $J$  using the Controllability Gramian of system (19) we have

$$J \leq \text{trace} \left( \tilde{C} \tilde{Y} \tilde{C}^T \right), \quad (20)$$

where  $\tilde{Y}$  satisfies the Lyapunov inequality

$$\tilde{A} \tilde{Y} + \tilde{Y} \tilde{A}^T + \tilde{B}_w W \tilde{B}_w^T \prec 0.$$

Recall that  $\tilde{D}$  must be zero!

Now introduce the matrix  $Z$  as in

$$Z \succ \tilde{C} \tilde{Y} \tilde{C}^T$$

so that

$$\text{trace}(Z) > \text{trace} \left( \tilde{C} \tilde{Y} \tilde{C}^T \right) \geq J.$$

### 11.3 Method I (congruence + change-of-variables)

Congruence transformation and change-of-variables are far from trivial. We will take small steps. First apply Schur complement to transform the analysis conditions into the more convenient form

$$\begin{bmatrix} \tilde{A}\tilde{Y} + \tilde{Y}\tilde{A}^T & \tilde{B} \\ \tilde{B}^T & -W^{-1} \end{bmatrix} \prec 0, \quad \begin{bmatrix} Z & \tilde{C}\tilde{Y} \\ \tilde{Y}\tilde{C}^T & \tilde{Y} \end{bmatrix} \succ 0, \quad \tilde{D} = 0.$$

We will apply congruence transformations of the form

$$\begin{bmatrix} \tilde{T}^T & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \tilde{A}\tilde{Y} + \tilde{Y}\tilde{A}^T & \tilde{B} \\ \tilde{B}^T & -W^{-1} \end{bmatrix} \begin{bmatrix} \tilde{T} & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} \tilde{T}^T \tilde{A}\tilde{Y}\tilde{T} + \tilde{T}^T \tilde{Y}\tilde{A}^T \tilde{T} & \tilde{T}^T \tilde{B} \\ \tilde{B}\tilde{T} & -W^{-1} \end{bmatrix} \prec 0,$$

$$\begin{bmatrix} I & 0 \\ 0 & \tilde{T}^T \end{bmatrix} \begin{bmatrix} Z & \tilde{C}\tilde{Y} \\ \tilde{Y}\tilde{C}^T & \tilde{Y} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \tilde{T} \end{bmatrix} = \begin{bmatrix} Z & \tilde{C}\tilde{Y}\tilde{T} \\ \tilde{T}^T \tilde{Y}\tilde{C}^T & \tilde{T}^T \tilde{Y}\tilde{T} \end{bmatrix} \succ 0,$$

where  $\tilde{T}$  is a matrix yet to be determined. From the above inequalities, LMIs are obtained if we are able to determine  $\tilde{T}$  such that the matrices

$$\tilde{T}^T \tilde{A}\tilde{Y}\tilde{T}, \quad \tilde{C}\tilde{Y}\tilde{T}, \quad \tilde{T}^T \tilde{B},$$

are affine functions of the design variables, perhaps after a change of variables.

## 11.4 The Congruence Transformation

Define the partitions associated with  $\tilde{Y}$  and its inverse

$$\tilde{Y} := \begin{bmatrix} X & U^T \\ U & \hat{X} \end{bmatrix}, \quad \tilde{Y}^{-1} := \begin{bmatrix} Y & V \\ V^T & \hat{Y} \end{bmatrix}.$$

Now define the transformation matrix  $\tilde{T}$  as

$$\tilde{T} := \begin{bmatrix} I & Y \\ 0 & V^T \end{bmatrix}.$$

Now verify that

$$\tilde{Y}\tilde{T} = \begin{bmatrix} X & I \\ U & 0 \end{bmatrix}$$

and

$$\begin{aligned} \tilde{T}^T \tilde{A} \tilde{Y} \tilde{T} &= \begin{bmatrix} AX + B_u(C_c U + D_c C_y X) & A + B_u D_c C_y \\ (\star) & YA + (VB_c + YB_u D_c) C_y \end{bmatrix}, \\ \tilde{C} \tilde{Y} \tilde{T} &= [C_z X + D_{zu}(C_c U + D_c C_y X) \quad C_z + D_{zu} D_c C_y], \\ \tilde{T}^T \tilde{B} &= \begin{bmatrix} B_w + B_u D_c D_{yw} \\ YB_w + (VB_c + YB_u D_c) D_{yw} \end{bmatrix}, \\ \tilde{T}^T \tilde{Y} \tilde{T} &= \begin{bmatrix} X & I \\ I & Y \end{bmatrix}, \end{aligned}$$

where

$$(\star) := VA_c U + YAX + VB_c C_y X + YB_u C_c U + YB_u D_c C_y X.$$

## 11.5 The Change-of-Variables

The change-of-variables

$$R := D_c,$$

$$L := C_c U + D_c C_y X,$$

$$F := V B_c + Y B_u D_c,$$

$$Q := V A_c U + Y A X + V B_c C_y X + Y B_u C_c U + Y B_u D_c C_y X = (\star)$$

transform

$$\tilde{T}^T \tilde{A} \tilde{Y} \tilde{T} = \begin{bmatrix} AX + B_u (C_c U + D_c C_y X) & A + B_u D_c C_y \\ (\star) & YA + (V B_c + Y B_u D_c) C_y \end{bmatrix},$$

$$\tilde{C} \tilde{Y} \tilde{T} = [C_z X + D_{zu} (C_c U + D_c C_y X) \quad C_z + D_{zu} D_c C_y],$$

$$\tilde{T}^T \tilde{B} = \begin{bmatrix} B_w + B_u D_c D_{yw} \\ Y B_w + (V B_c + Y B_u D_c) D_{yw} \end{bmatrix},$$

$$\tilde{T}^T \tilde{Y} \tilde{T} = \begin{bmatrix} X & I \\ I & Y \end{bmatrix},$$

into

$$\tilde{T}^T \tilde{A} \tilde{X} \tilde{T} = \begin{bmatrix} AX + B_u L & A + B_u R C_y \\ Q & YA + F C_y \end{bmatrix},$$

$$\tilde{C} \tilde{X} \tilde{T} = [C_z X + D_{zu} L \quad C_z + D_{zu} R C_y],$$

$$\tilde{T}^T \tilde{B} = \begin{bmatrix} B_w + B_u R D_{yw} \\ Y B_w + F D_{yw} \end{bmatrix},$$

$$\tilde{T}^T \tilde{X} \tilde{T} = \begin{bmatrix} X & I \\ I & Y \end{bmatrix},$$

which are affine in the *synthesis variables*  $Q, L, F, R$ .

## 11.6 The LMIs

Those terms can be replaced in the analysis conditions providing the LMI synthesis conditions

$$\begin{aligned}
 0 &\prec \begin{bmatrix} \tilde{T}^T & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \tilde{A}\tilde{Y} + \tilde{Y}\tilde{A}^T & \tilde{B} \\ \tilde{B}^T & -W^{-1} \end{bmatrix} \begin{bmatrix} \tilde{T} & 0 \\ 0 & I \end{bmatrix} \\
 &= \begin{bmatrix} AX + B_u L + XA^T + L^T B_u^T & A + B_u R C_y + Q^T & B_w + B_u R D_{yw} \\ (\bullet)^T & A^T Y + C_y^T F^T + Y A + F C_y & Y B_w + F D_{yw} \\ (\bullet)^T & (\bullet)^T & -W^{-1} \end{bmatrix},
 \end{aligned}$$

and

$$\begin{aligned}
 0 &\prec \begin{bmatrix} I & 0 \\ 0 & \tilde{T}^T \end{bmatrix} \begin{bmatrix} Z & \tilde{C}\tilde{Y} \\ \tilde{Y}\tilde{C}^T & \tilde{Y} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \tilde{T}^T \end{bmatrix} \\
 &= \begin{bmatrix} Z & C_z X + D_{zu} L & C_z + D_{zu} R C_y \\ (\bullet)^T & X & I \\ (\bullet)^T & (\bullet)^T & Y \end{bmatrix},
 \end{aligned}$$

and the condition that  $\tilde{D} = 0$  into

$$D_{zw} + D_{zu} R D_{yw} = 0$$

which is linear on  $R$ .

Notice that the above relations do not depend on  $V$ ,  $U$  or  $\hat{X}$  and  $\hat{Y}$ .

## 11.7 A Closer Look at the Change-of-Variables

The change-of-variables can be put in the more compact form

$$\begin{bmatrix} Q & F \\ L & R \end{bmatrix} := \begin{bmatrix} V & YB_u \\ 0 & I \end{bmatrix} \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} \begin{bmatrix} U & 0 \\ C_y X & I \end{bmatrix} + \begin{bmatrix} Y \\ 0 \end{bmatrix} A \begin{bmatrix} X & 0 \end{bmatrix}.$$

The first important point to establish is whether this transformation is invertible. The simplest case is that of  $V$  and  $U$  square which, if nonsingular lead to

$$\begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} = \begin{bmatrix} V^{-1} & -V^{-1}YB_u \\ 0 & I \end{bmatrix} \begin{bmatrix} Q - YAX & F \\ L & R \end{bmatrix} \begin{bmatrix} U^{-1} & 0 \\ -C_y XU^{-1} & I \end{bmatrix}.$$

This suggest the following questions to be answered:

1. When are  $V$  and  $U$  square?
2. If  $V$  and  $U$  square, are they nonsingular?
3. What if they are not square?

The answer to these questions are

1.  $V$  and  $U$  are square when  $n_c = n$ , i.e., the controller and plant have the same order (recall state observers);
2. They are also nonsingular because

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \succ 0$$

which implies that  $X \succ 0$  and  $Y - X^{-1} \succ 0$ . Consequently that

$$(Y - X^{-1})X = YX - I$$

is nonsingular. Because

$$I = \tilde{Y}^{-1}\tilde{Y} = \begin{bmatrix} Y & V \\ V^T & \hat{Y} \end{bmatrix} \begin{bmatrix} X & U^T \\ U & \hat{X} \end{bmatrix}$$

we have that

$$YX + VU = I \quad \implies \quad VU = I - YX$$

so that both  $U$  and  $V$  must be nonsingular.



3. What if they are not square? The next simplest case is when  $n_c > n$  in which case  $V \in \mathbb{R}^{n \times n_c}$  and  $U \in \mathbb{R}^{n_c \times n}$  and

$$YX + VU = I$$

can be satisfied by some  $U$  and  $V$  such that  $VU$  is nonsingular. Note that in the case it can be satisfied for some  $n_c > n$  it can also be satisfied with  $n_c = n$ . That is, there is not need to have a controller of order higher than the plant!

The complicated case is when  $n_c < n$ . In this case

$$VU = I - YX$$

is never nonsingular so that  $Y - X^{-1} \neq 0$ . Indeed one can show that the strict inequalities must be relaxed, for instance

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \succeq 0, \quad X \succ 0, \quad Y \succ 0, \quad \text{rank}(YX - I) = n_c.$$

In the case of static output feedback ( $n_c = 0$ ) this becomes

$$X = Y^{-1} \succ 0.$$

We do not like rank's that much :(.

## 11.8 A Matrix Completion Problem

A final question is that of whether there exists  $\hat{X}$  and  $\hat{Y}$  so that

$$I = \tilde{Y}^{-1}\tilde{Y} = \begin{bmatrix} Y & V \\ V^T & \hat{Y} \end{bmatrix} \begin{bmatrix} X & U^T \\ U & \hat{X} \end{bmatrix}$$

once variables  $X$  and  $Y$  satisfying the LMIs and  $U$  and  $V$  satisfying

$$VU = I - YX$$

have been computed. Let us focus on the case  $n_c = n$ . In this case, either  $U$  or  $V$  can be set arbitrarily to any nonsingular matrix. For instance

$$U = X, \quad V = (I - YX)U^{-1} = X^{-1} - Y.$$

For such a choice

$$\begin{aligned} I = \tilde{Y}^{-1}\tilde{Y} &= \begin{bmatrix} Y & V \\ V^T & \hat{Y} \end{bmatrix} \begin{bmatrix} X & U^T \\ U & \hat{X} \end{bmatrix} \\ &= \begin{bmatrix} I & YX + (X^{-1} - Y)\hat{X} \\ (X^{-1} - Y)X + \hat{Y}X & (X^{-1} - Y)X + \hat{Y}\hat{X} \end{bmatrix} \end{aligned}$$

which implies that

$$\begin{aligned} YX + (X^{-1} - Y)\hat{X} &= 0, \\ (X^{-1} - Y)X + \hat{Y}X &= 0, \\ (X^{-1} - Y)X + \hat{Y}\hat{X} &= I. \end{aligned}$$

Using the fact that  $X \succ 0$  we can solve the second equation for  $\hat{Y}$

$$\hat{Y} = Y - X^{-1}$$

and the third for  $\hat{X}$

$$\hat{X} = X - (X^{-1} - Y)^{-1}.$$

One can verify that the first equation is automatically satisfied.

With lots of applications of the Lemma of the Matrix Inverse one can show that the above is true for any suitable choice of  $U$  and  $V$ .

## 11.9 Summary: Output Feedback Control

The full order dynamic output feedback controller

$$\begin{aligned}\dot{x}_c &= A_c x_c + B_c y, & x_c(0) &= 0, & x_c &\in \mathbb{R}^n \\ u &= C_c x_c + D_c y,\end{aligned}$$

computed as

$$\begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} = \begin{bmatrix} V^{-1} & -V^{-1}YB_u \\ 0 & I \end{bmatrix} \begin{bmatrix} Q - YAX & F \\ L & R \end{bmatrix} \begin{bmatrix} U^{-1} & 0 \\ -C_y XU^{-1} & I \end{bmatrix},$$

where  $X, Y \in \mathbb{S}^n$ ,  $L \in \mathbb{R}^{m \times n}$ ,  $F \in \mathbb{R}^{n \times q}$ ,  $Q \in \mathbb{R}^{n \times n}$ ,  $R \in \mathbb{R}^{m \times q}$  and  $Z \in \mathbb{S}^r$  solve the SDP

$$\begin{aligned} & \inf_{X, L, Z} \text{trace}(Z) \\ \text{s.t.} & \begin{bmatrix} \left\{ \begin{array}{l} AX + XA^T + \\ B_u L + L^T B_u^T \end{array} \right\} & A + B_u R C_y + Q^T & B_w + B_u R D_{yw} \\ (\bullet)^T & \left\{ \begin{array}{l} A^T Y + Y A + \\ F C_y + C_y^T F^T \end{array} \right\} & Y B_w + F D_{yw} \\ (\bullet)^T & (\bullet)^T & -W^{-1} \end{bmatrix} \prec 0, \\ & \begin{bmatrix} Z & C_z X + D_{zu} L & C_z + D_{zu} R C_y \\ (\bullet)^T & X & I \\ (\bullet)^T & (\bullet)^T & Y \end{bmatrix} \succ 0, \\ & D_{zw} + D_{zu} R D_{yw} = 0 \end{aligned}$$

and  $U, V \in \mathbb{R}^{n \times n}$  are such that

$$YX + VU = I$$

is the optimal solution to the problem of stabilizing the closed loop system and minimizing the cost function

$$J := \lim_{t \rightarrow \infty} E [z(t)^T z(t)] \leq \text{trace}(Z)$$

for the LTI system

$$\begin{aligned}\dot{x}(t) &= Ax(t) + B_u u(t) + B_w w(t), & x(0) &= 0, & x &\in \mathbb{R}^n \\ z(t) &= C_z x(t) + D_{zw} w(t) + D_{zu} u(t)\end{aligned}$$

under the assumptions

1.  $(A, B_u)$  stabilizable,
2.  $(A, C_y)$  detectable,
3.  $D_{zu}^T D_{zu} \succ 0$ ,
4.  $D_{yw} W D_{yw}^T \succ 0$ ,
5.  $w(t)$  is a Gaussian zero mean white noise with variance  $W \succ 0$ .