10 Non-Classic State Feedback Control Design

10.1 Decentralized LQR Control

Problem: Given the LTI system

$$\begin{aligned}\dot{x}(t) &= Ax(t) + B_u u(t) + B_w w(t), \quad x(0) = 0\\ z(t) &= C_z x(t) + D_{zu} u(t).\end{aligned}$$

where

$$x(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_N(t) \end{pmatrix}, \qquad x_i \in \mathbb{R}^{n_i}, \qquad \sum_{i=1}^{N} n_i = n,$$

compute a decentralized state feedback controller

$$u_i(t) = K_i\, x_i(t), \qquad u(t) = \begin{pmatrix} u_1(t) \\ \vdots \\ u_N(t) \end{pmatrix}$$

that stabilizes the closed loop system and minimizes

$$J := \lim_{t\to\infty} E\left[z(t)^T z(t)\right]$$

Assumptions:

a) $w(t)$ is a Gaussian white noise vector with zero mean and covariance $W \succ 0$.

Solution: We do not know of a complete solution to this problem. Here we will be happy with a sufficient design (conservative). The key is to note that

$$u(t) = \begin{pmatrix} u_1(t) \\ \vdots \\ u_N(t) \end{pmatrix} = \begin{pmatrix} K_1\, x_1(t) \\ \vdots \\ K_N\, x_N(t) \end{pmatrix} = K_D x(t)$$

where

$$K_D = \begin{bmatrix} K_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & K_N \end{bmatrix}$$

is block-diagonal!

Using the congruence+change-of-variables the LQR problem can be recast as the SDP

$$\begin{aligned}\min_{X\in\mathbb{S}^n, L\in\mathbb{R}^{m\times n}, Z\in\mathbb{S}^r} \quad & \mathrm{trace}(ZW) \\ \text{s.t.} \quad & \begin{bmatrix} AX + XA^T + B_u L + L^T B_u^T & XC_z^T + L^T D_{zu}^T \\ C_z X + D_{zu} L & -I \end{bmatrix} \preceq 0, \\ & \begin{bmatrix} Z & B_w^T \\ B_w & X \end{bmatrix} \succeq 0, \quad X \succ 0.\end{aligned}$$

where $K = LX^{-1}$. Now note that under the additional constraint $X = X_D$ and $L = L_D$, i.e. X and L are block-diagonal then

$$X = X_D \text{ and } L = L_D \qquad \Longrightarrow \qquad L_D X_D^{-1} = K_D.$$

Note that the original problem does not require $X = X_D$!
Also note that

$$\mathrm{trace}(ZW) \geq J = \lim_{t\to\infty} E\left[z(t)^T z(t)\right]$$

Recipe II (existence condition) does not provide any solution to this problem. (why?)

10.1.1 Summary: Decentralized LQR Control

The decentralized state feedback controller

$$u(t) = K_D x(t), \qquad\qquad K_D = L_D X_D^{-1}$$

where $X \in \mathbb{S}^n$, $L \in \mathbb{R}^{m\times n}$ and $Z \in \mathbb{S}^r$ solve the SDP

$$\begin{aligned}
\min_{X\in\mathbb{S}^n, L\in\mathbb{R}^{m\times n}, Z\in\mathbb{S}^r} \quad & \operatorname{trace}(ZW) \\
\text{s.t.} \quad & \begin{bmatrix} AX + XA^T + B_u L + L^T B_u^T & XC_z^T + L^T D_{zu}^T \\ C_z X + D_{zu} L & -I \end{bmatrix} \preceq 0, \\
& \begin{bmatrix} Z & B_w^T \\ B_w & X \end{bmatrix} \succeq 0, \quad X \succ 0, \\
& X = X_D, \qquad L = L_D,
\end{aligned}$$

is a suboptimal solution to the problem of stabilizing the closed loop system and minimizing the cost function

$$J := \lim_{t\to\infty} E\left[z(t)^T z(t)\right] \leq \operatorname{trace}(ZW)$$

for the LTI system

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + B_u u(t) + B_w w(t), \quad x(0) = 0 \\
z(t) &= C_z x(t) + D_{zu} u(t)
\end{aligned}$$

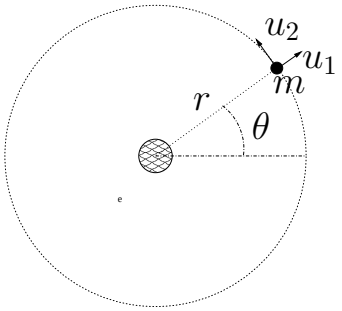
under the assumptions

a) $D_{zu}^T D_{zu} \succ 0$;

b) (A, B_u) stabilizable;

c) $w(t)$ is a Gaussian white noise vector with zero mean and covariance $W \succ 0$.

10.2 Example: satellite in circular orbit



Satellite of mass m with thrust in the radial direction u_1 and in the tangential direction u_2. Continuing...

$$m(\ddot{r} - r\dot{\theta}^2) = u_1 - \frac{km}{r^2} + w_1,$$
$$m(2\dot{r}\dot{\theta} + r\ddot{\theta}) = u_2 + w_2,$$

where w_1 and w_2 are independent white noise disturbances with variances δ_1 and δ_2.
As before, putting in state space and linearizing around

$$\bar{x}(t) = \begin{pmatrix} \bar{r} \\ \bar{\omega}t \\ 0 \\ \bar{\omega} \end{pmatrix}, \qquad \bar{u} = \begin{pmatrix} \bar{u}_1 \\ \bar{u}_2 \end{pmatrix} = 0, \qquad \bar{w} = \begin{pmatrix} \bar{w}_1 \\ \bar{w}_2 \end{pmatrix} = 0, \qquad \bar{\omega} = \sqrt{k/\bar{r}^3}$$

one obtains the linearized system

$$\dot{x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 3\,\bar{\omega}^2 & 0 & 0 & 2\,\bar{r}\bar{\omega} \\ 0 & 0 & -2\,\bar{\omega}/\bar{r} & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1/m & 0 \\ 0 & 1/(m\bar{r}) \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1/m & 0 \\ 0 & 1/(m\bar{r}) \end{bmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.$$

Problem: Given

$$m = 100 \text{ kg}, \qquad \bar{r} = R + 300 \text{ km}, \qquad \bar{k} = GM$$

where $G \approx 6.673 \times 10^{-11}$ N m^2/kg^2 is the universal gravitational constant, and $M \approx 5.98 \times 10^{24}$ kg and $R \approx 6.37 \times 10^3$ km are the mass and radius of the earth. If the variances δ_1 and δ_2 are $0.1N$ find solutions to the LQR control problem where

$$Q = I, \qquad R = \rho I,$$

and u_1 only feedbacks r and $\dot{r}$, u_2 only feedbacks θ and $\dot{\theta}$.

In order for this problem to be cast as a 'decentralized' control problem we need to reorder the state. This can be done through a similarity transformation

$$\tilde{x} = \begin{pmatrix} r \\ \dot{r} \\ \theta \\ \dot{\theta} \end{pmatrix} = Tx = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} r \\ \theta \\ \dot{r} \\ \dot{\theta} \end{pmatrix}$$

and partition the state

$$\tilde{x} = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix}, \qquad \tilde{x}_1 = \begin{pmatrix} r \\ \dot{r} \end{pmatrix}, \qquad \tilde{x}_2 = \begin{pmatrix} \theta \\ \dot{\theta} \end{pmatrix}$$

so that the decentralized control

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \qquad u_1 = K_1 \, \tilde{x}_1, \qquad u_2 = K_2 \, \tilde{x}_2$$

have the desired structure.

```
% MAE 280 B - Linear Control Design
% Mauricio de Oliveira
%
% LMI LQR Control
%
m = 100;                % 100 kg
r = 300E3;              % 300 km
R = 6.37E6;             % 6.37 10^3 km
G = 6.673E-11;          % 6.673 N m^2/kg^2
M = 5.98E24;            % 5.98 10^24 kg
k = G * M;              % gravitational force constant
w = sqrt(k/((R+r)^3)); % angular velocity (rad/s)
v = w * (R + r);        % "ground" velocity (m/s)

% linearized system matrices
A = [0 0 1 0; 0 0 0 1; 3*w^2 0 0 2*(r+R)*w; 0 0 -2*w/(r+R) 0];
Bu = [0 0; 0 0; 1/m 0; 0 1/(m*r)];
Bw = [0 0; 0 0; 1/m 0; 0 1/(m*r)];

% noise variances
W = 0.1 * eye(2)
W =
   1.0000e-01            0
             0   1.0000e-01

% scale
T = diag([1 r 1 r])
T =
           1           0           0           0
```

```
           0      300000           0           0
           0           0           1           0
           0           0           0      300000

% similarity transformation
At = T * A / T
At =
            0            0   1.0000e+00           0
            0            0           0   1.0000e+00
   4.0343e-06            0           0   5.1565e-02
            0            0  -1.0432e-04           0
But = T * Bu
But =
            0            0
            0            0
   1.0000e-02            0
            0   1.0000e-02
Bwt = T * Bw
Bwt =
            0            0
            0            0
   1.0000e-02            0
            0   1.0000e-02

% another similarity to change state
T = [1 0 0 0; 0 0 1 0; 0 1 0 0; 0 0 0 1]
T =
     1     0     0     0
     0     0     1     0
     0     1     0     0
     0     0     0     1
Att = T * At / T
Att =
            0   1.0000e+00           0           0
   4.0343e-06            0           0   5.1565e-02
            0            0           0   1.0000e+00
            0  -1.0432e-04           0           0
Butt = T * But
Butt =
            0            0
   1.0000e-02            0
            0            0
            0   1.0000e-02
Bwtt = T * Bwt
Bwtt =
            0            0
   1.0000e-02            0
            0            0
            0   1.0000e-02

% optimal LQR state feedback control (using u1 and u2)
n = size(Att,1);
m = size(Butt,2);
```

```
% LQR solution
Cz = [eye(n); zeros(m,n)];
Dzu = [zeros(n,m); eye(m)];

[K,X,S] = lqr(Att, Butt, Cz' * Cz, Dzu' * Dzu);
Klqr = - K
Klqr =
  -9.8444e-01  -1.3843e+01   1.7795e-01  -2.4919e+00
  -1.7795e-01  -2.4919e+00  -9.8404e-01  -1.4741e+01

% LMI solution with decentralization
% declare variables
X = sdpvar(n,n,'symmetric');
Z = sdpvar(m,m,'symmetric');
L = sdpvar(m,n);

% declare LMIs
LMI1 = [Att*X+X*Att'+Butt*L+L'*Butt', X*Cz'+L'*Dzu';...
        Cz*X+Dzu*L, -eye(m+n)];
LMI2 = [Z, Bwtt'; Bwtt, X];

% Structural constraints
Xmask = [zeros(2,2) ones(2,2); ones(2,2) zeros(2,2)]
Xmask =
     0     0     1     1
     0     0     1     1
     1     1     0     0
     1     1     0     0
Lmask = [zeros(1,2) ones(1,2); ones(1,2) zeros(1,2)]
Lmask =
     0     0     1     1
     1     1     0     0

LMI = set(LMI1 < 0) + set(LMI2 > 0) + set(Xmask.*X == 0) + set(Lmask.*L == 0);
options = sdpsettings('solver','sedumi');
solution =solvesdp(LMI,trace(W*Z),options)
SeDuMi 1.1R3 by AdvOL, 2006 and Jos F. Sturm, 1998-2003.
Alg = 2: xz-corrector, theta = 0.250, beta = 0.500
Put 24 free variables in a quadratic cone
eqs m = 21, order n = 19, dim = 162, blocks = 4
nnz(A) = 77 + 0, nnz(ADA) = 393, nnz(L) = 207
 it :     b*y       gap    delta  rate   t/tP*  t/tD*   feas cg cg  prec
  0 :            5.84E+00 0.000
  1 :  -1.21E-01 1.26E+00 0.000 0.2164 0.9000 0.9000   2.32  1  1  1.5E+00
  2 :  -3.23E-02 2.56E-01 0.000 0.2028 0.9000 0.9000   1.54  1  1  7.1E-01
  3 :  -6.52E-03 5.86E-02 0.000 0.2284 0.9000 0.9000   1.23  1  1  5.4E-01
  4 :  -3.90E-03 1.63E-02 0.000 0.2775 0.9000 0.9000   1.04  1  1  7.3E-01
  5 :  -5.48E-03 4.82E-03 0.000 0.2964 0.9000 0.9000   0.56  1  1  2.8E-01
  6 :  -7.91E-03 1.61E-03 0.000 0.3348 0.9000 0.9000   0.14  1  1  1.1E-01
  7 :  -1.13E-02 5.44E-04 0.000 0.3370 0.9000 0.9000   0.14  1  1  7.1E-03
  8 :  -1.47E-02 2.05E-04 0.000 0.3770 0.9000 0.9000   0.10  1  1  6.6E-04
  9 :  -1.92E-02 6.99E-05 0.000 0.3412 0.9000 0.9000   0.17  1  1  3.6E-04
 10 :  -2.28E-02 2.80E-05 0.000 0.4005 0.9000 0.9000   0.13  1  1  2.4E-04
 11 :  -2.76E-02 9.32E-06 0.000 0.3328 0.9000 0.9000   0.34  1  1  1.1E-04
```

```
 12 :  -3.09E-02 3.44E-06 0.000 0.3686 0.9000 0.9000   0.39  1  1  5.9E-05
 13 :  -3.35E-02 9.53E-07 0.000 0.2775 0.9000 0.9000   0.64  1  1  1.9E-05
 14 :  -3.46E-02 2.27E-07 0.000 0.2379 0.9000 0.9000   0.79  1  1  5.2E-06
 15 :  -3.50E-02 1.23E-08 0.000 0.0542 0.9900 0.9900   0.94  1  1  2.9E-07
 16 :  -3.50E-02 2.75E-09 0.000 0.2234 0.9000 0.9000   0.97  1  1  6.6E-08
 17 :  -3.50E-02 2.11E-11 0.056 0.0077 0.9990 0.9990   1.00  1  1  5.1E-10

iter seconds digits       c*x               b*y
 17      0.1   Inf -3.5011343878e-02 -3.5011340045e-02
|Ax-b| =   3.0e-10, [Ay-c]_+ =   1.4E-10, |x|=  2.8e+02, |y|=  2.9e-01

Detailed timing (sec)
   Pre          IPM          Post
1.000E-02    1.300E-01    0.000E+00
Max-norms: ||b||=1.000000e-01, ||c|| = 1,
Cholesky |add|=0, |skip| = 0, ||L.L|| = 391.968.
solution =
    yalmiptime: 6.2184e-02
    solvertime: 1.4307e-01
          info: 'No problems detected (SeDuMi-1.1)'
       problem: 0
        dimacs: [2.7569e-10 0 0 7.1088e-11 -3.5825e-09 -3.5397e-09]

% Construct K and check closed loop stability and performance
double(L)
ans =
   4.4058e-07  -1.0000e-02  -2.2262e-16  -1.0398e-16
  -8.9748e-17  -7.0191e-17  -1.1328e-07  -1.0000e-02
double(X)
ans =
   1.3231e-01  -8.7973e-03  -5.8520e-13  -1.0460e-11
  -8.7973e-03   1.1741e-03  -4.3354e-12  -1.0041e-10
  -5.8520e-13  -4.3354e-12   7.9953e-02  -4.4339e-03
  -1.0460e-11  -1.0041e-10  -4.4339e-03   8.0025e-04
Ld = double(L) - double(L) .* Lmask
Ld =
   4.4058e-07  -1.0000e-02            0            0
            0            0  -1.1328e-07  -1.0000e-02
Xd = double(X) - double(X) .* Xmask
Xd =
   1.3231e-01  -8.7973e-03            0            0
  -8.7973e-03   1.1741e-03            0            0
            0            0   7.9953e-02  -4.4339e-03
            0            0  -4.4339e-03   8.0025e-04
Klmi = Ld / Xd
Klmi =
  -1.1285e+00  -1.6973e+01            0            0
            0            0  -1.0003e+00  -1.8039e+01
Acl = Att + Butt*Klmi;
eig(Acl)
ans =
  -8.5040e-02 + 6.3757e-02i
  -8.5040e-02 - 6.3757e-02i
  -9.0017e-02 + 4.3439e-02i
```

```
  -9.0017e-02 - 4.3439e-02i
Y = lyap(Acl, Bwtt * W * Bwtt')
Y =
   2.7272e-03  -1.2536e-19   3.8153e-04   1.3926e-06
  -1.2536e-19   3.0695e-05  -1.3926e-06   4.0678e-06
   3.8153e-04  -1.3926e-06   2.7707e-03  -1.8974e-19
   1.3926e-06   4.0678e-06  -1.8974e-19   2.7716e-05
sqrt(trace(Y))
ans =
   7.4540e-02
sqrt(diag(Y))
ans =
   5.2223e-02
   5.5403e-03
   5.2637e-02
   5.2646e-03

% Try stripping lqr gain
Klqr
Klqr =
  -9.8444e-01  -1.3843e+01   1.7795e-01  -2.4919e+00
  -1.7795e-01  -2.4919e+00  -9.8404e-01  -1.4741e+01
Klqrd = double(Klqr) - double(Klqr) .* Lmask
Klqrd =
  -9.8444e-01  -1.3843e+01            0            0
            0            0  -9.8404e-01  -1.4741e+01
Acl = Att + Butt*Klqrd;
eig(Acl)
ans =
  -6.9537e-02 + 7.0746e-02i
  -6.9537e-02 - 7.0746e-02i
  -7.3382e-02 + 6.6749e-02i
  -7.3382e-02 - 6.6749e-02i
Y = lyap(Acl, Bwtt * W * Bwtt')
Y =
   3.9016e-03   3.3881e-20   6.2032e-04   1.7378e-12
   3.3881e-20   3.8393e-05  -1.7378e-12   6.1042e-06
   6.2032e-04  -1.7378e-12   3.4465e-03   1.2875e-19
   1.7378e-12   6.1042e-06   1.2875e-19   3.3915e-05
sqrt(trace(Y))
ans =
   8.6142e-02
sqrt(diag(Y))
ans =
   6.2463e-02
   6.1962e-03
   5.8707e-02
   5.8236e-03

% LQR solution
Cz = [eye(n); zeros(m,n)];
Dzu = 1e4*[zeros(n,m); eye(m)];
w
w =
```

```
   1.1596e-03
[K,X,S] = lqr(Att, Butt, Cz' * Cz, Dzu' * Dzu);
Klqr = - K
Klqr =
  -1.9615e-05  -6.5712e-03   3.8376e-05  -5.2724e-02
  -1.5950e-04  -5.2724e-02   9.2343e-05  -7.2284e-01

% LMI solution with decentralization
% declare variables
X = sdpvar(n,n,'symmetric');
Z = sdpvar(m,m,'symmetric');
L = sdpvar(m,n);

% declare LMIs
LMI1 = [Att*X+X*Att'+Butt*L+L'*Butt', X*Cz'+L'*Dzu';...
        Cz*X+Dzu*L, -eye(m+n)];
LMI2 = [Z, Bwtt'; Bwtt, X];

% Structural constraints
Xmask = [zeros(2,2) ones(2,2); ones(2,2) zeros(2,2)]
Xmask =
     0     0     1     1
     0     0     1     1
     1     1     0     0
     1     1     0     0
Lmask = [zeros(1,2) ones(1,2); ones(1,2) zeros(1,2)]
Lmask =
     0     0     1     1
     1     1     0     0

LMI = set(LMI1 < 0) + set(LMI2 > 0) + set(Xmask.*X == 0) + set(Lmask.*L == 0);
options = sdpsettings('solver','sedumi');
solution =solvesdp(LMI,trace(W*Z),options)
SeDuMi 1.1R3 by AdvOL, 2006 and Jos F. Sturm, 1998-2003.
Alg = 2: xz-corrector, theta = 0.250, beta = 0.500
Put 24 free variables in a quadratic cone
eqs m = 21, order n = 19, dim = 162, blocks = 4
nnz(A) = 77 + 0, nnz(ADA) = 393, nnz(L) = 207
 it :     b*y       gap    delta  rate   t/tP*  t/tD*   feas cg cg  prec
  0 :            5.84E+00 0.000
  1 :  -1.21E-01 1.26E+00 0.000 0.2164 0.9000 0.9000   2.32  1  1  1.5E+00
  2 :  -3.23E-02 2.57E-01 0.000 0.2032 0.9000 0.9000   1.54  1  1  7.2E-01
  3 :  -6.61E-03 5.91E-02 0.000 0.2300 0.9000 0.9000   1.23  1  1  5.5E-01
  4 :  -3.96E-03 1.65E-02 0.000 0.2789 0.9000 0.9000   1.03  1  1  7.4E-01
  5 :  -5.66E-03 4.92E-03 0.000 0.2988 0.9000 0.9000   0.53  1  1  2.9E-01
  6 :  -8.40E-03 1.62E-03 0.000 0.3282 0.9000 0.9000   0.09  1  1  1.1E-01
  7 :  -1.25E-02 5.47E-04 0.000 0.3382 0.9000 0.9000   0.04  1  1  1.2E-03
  8 :  -1.73E-02 1.95E-04 0.000 0.3565 0.9000 0.9000  -0.04  1  1  8.3E-04
  9 :  -2.43E-02 6.65E-05 0.000 0.3411 0.9000 0.9000  -0.03  1  1  5.2E-04
 10 :  -3.25E-02 2.33E-05 0.000 0.3502 0.9000 0.9000  -0.15  1  1  4.0E-04
 11 :  -4.64E-02 7.77E-06 0.000 0.3337 0.9000 0.9000  -0.07  1  1  2.4E-04
 12 :  -6.39E-02 2.40E-06 0.000 0.3090 0.9000 0.9000  -0.18  1  1  1.8E-04
 13 :  -9.04E-02 7.90E-07 0.000 0.3290 0.9000 0.9000  -0.10  1  1  1.1E-04
 14 :  -1.24E-01 2.49E-07 0.000 0.3156 0.9000 0.9000  -0.19  1  1  7.9E-05
```

```
 15 :  -1.76E-01 8.09E-08 0.000 0.3248 0.9000 0.9000  -0.12  1  1  4.9E-05
 16 :  -2.45E-01 2.46E-08 0.000 0.3033 0.9000 0.9000  -0.20  1  1  3.5E-05
 17 :  -3.47E-01 8.05E-09 0.000 0.3278 0.9000 0.9000  -0.13  1  1  2.2E-05
 18 :  -4.79E-01 2.55E-09 0.000 0.3174 0.9000 0.9000  -0.20  1  1  1.6E-05
 19 :  -6.80E-01 8.41E-10 0.000 0.3294 0.9000 0.9000  -0.13  1  1  1.0E-05
 20 :  -9.47E-01 2.61E-10 0.000 0.3101 0.9000 0.9000  -0.21  1  1  7.2E-06
 21 :  -1.34E+00 8.68E-11 0.000 0.3329 0.9000 0.9000  -0.14  1  1  4.6E-06
 22 :  -1.87E+00 2.74E-11 0.000 0.3153 0.9000 0.9000  -0.21  1  1  3.3E-06
 23 :  -2.64E+00 9.19E-12 0.000 0.3357 0.9000 0.9000  -0.14  4  4  2.1E-06
 24 :  -3.71E+00 2.85E-12 0.000 0.3099 0.9000 0.9000  -0.21  5  5  1.5E-06
 25 :  -5.24E+00 9.65E-13 0.000 0.3388 0.9000 0.9000  -0.14  5  5  9.8E-07
 26 :  -7.37E+00 2.99E-13 0.000 0.3100 0.9000 0.9000  -0.21  5  5  7.0E-07
 27 :  -1.04E+01 1.02E-13 0.000 0.3416 0.9000 0.9000  -0.14  6  6  4.5E-07
 28 :  -1.48E+01 3.12E-14 0.000 0.3055 0.9000 0.9000  -0.21  6  6  3.2E-07
 29 :  -2.09E+01 1.08E-14 0.000 0.3448 0.9000 0.9000  -0.13  6  6  2.1E-07
 30 :  -2.91E+01 3.48E-15 0.000 0.3231 0.9000 0.9000  -0.21  6  6  1.5E-07
 31 :  -3.06E+01 3.00E-15 0.000 0.8610 0.9000 0.9000  -0.15  6  6  1.5E-07
 32 :  -3.84E+01 1.53E-15 0.000 0.5109 0.9000 0.9000  -0.17  6  6  1.1E-07
 33 :  -4.19E+01 1.17E-15 0.000 0.7647 0.9000 0.9000  -0.19  6  6  1.1E-07
Run into numerical problems.

iter seconds digits       c*x               b*y
 33      0.4   Inf -4.4375958352e+01 -4.1858430640e+01
|Ax-b| =   1.5e-07, [Ay-c]_+ =   2.9E-08, |x|=  1.2e+09, |y|=  3.1e+02

Detailed timing (sec)
   Pre          IPM          Post
0.000E+00    3.900E-01    0.000E+00
Max-norms: ||b||=1.000000e-01, ||c|| = 1,
Cholesky |add|=3, |skip| = 0, ||L.L|| = 500000.
solution =
    yalmiptime: 3.4341e-02
    solvertime: 4.0656e-01
          info: 'Numerical problems (SeDuMi-1.1)'
       problem: 4
        dimacs: [1.3558e-07 0 0 1.4401e-08 -2.8859e-02 -2.8859e-02]

% Construct K and check closed loop stability and performance
double(L)
ans =
  -2.2880e-16  -1.0000e-10  -9.9458e-19   7.8128e-19
  -5.2563e-18  -2.8990e-18  -1.2387e-16  -1.0000e-10
double(X)
ans =
   1.1629e-02  -9.5326e-05   3.0199e-13  -8.6117e-11
  -9.5326e-05   1.5550e-06   4.3864e-11  -1.1107e-08
   3.0199e-13   4.3864e-11   1.5769e-03  -6.4405e-06
  -8.6117e-11  -1.1107e-08  -6.4405e-06   4.3448e-07
Ld = double(L) - double(L) .* Lmask
Ld =
  -2.2880e-16  -1.0000e-10            0            0
            0            0  -1.2387e-16  -1.0000e-10
Xd = double(X) - double(X) .* Xmask
Xd =
```

```
   1.1629e-02  -9.5326e-05            0            0
  -9.5326e-05   1.5550e-06            0            0
            0            0   1.5769e-03  -6.4405e-06
            0            0  -6.4405e-06   4.3448e-07
Klmi = Ld / Xd
Klmi =
  -1.0597e-06  -1.2927e-04            0            0
            0            0  -1.0006e-06  -2.4499e-04
Acl = Att + Butt*Klmi;
eig(Acl)
ans =
  -5.2920e-06 + 1.1808e-03i
  -5.2920e-06 - 1.1808e-03i
   1.7338e-04
  -1.6654e-04

% Try stripping lqr gain
Klqr
Klqr =
  -1.9615e-05  -6.5712e-03   3.8376e-05  -5.2724e-02
  -1.5950e-04  -5.2724e-02   9.2343e-05  -7.2284e-01
Klqrd = double(Klqr) - double(Klqr) .* Lmask
Klqrd =
  -1.9615e-05  -6.5712e-03            0            0
            0            0   9.2343e-05  -7.2284e-01
Acl = Att + Butt*Klqrd;
eig(Acl)
ans =
  -6.4410e-03
  -2.6162e-03
   1.6344e-03
   1.2869e-04

diary off
```

10.3 Decentralized LQR Control — Dual form

10.3.1 LQR Control — Dual form

When we converted the LQR problem to LMIs we worked with the cost function

$$J = \text{trace}\left(P_o B_w W B_w^T\right)$$

where P_o is the observability Gramian

$$(A + B_u K)^T P_o + P_o(A + B_u K) + (C_z + D_{zu} K)^T (C_z + D_{zu} K) = 0.$$

An alternative form is provided by using the dual cost function

$$J = \text{trace}\left[(C_z + D_{zu} K) P_c (C_z + D_{zu} K)^T\right]$$

where P_c is the controllability Gramian

$$(A + B_u K) P_c + P_c (A + B_u K)^T + B_w W B_w^T = 0.$$

As before, use the Comparison Lemma to conclude that for any K such that $(A + B_u K)$ is Hurwitz then

$$X \succeq P_c \qquad \Longrightarrow \qquad \text{trace}\left[(C_z + D_{zu} K) X (C_z + D_{zu} K)^T\right] \geq J$$

where X satisfies the inequality

$$(A + B_u K) X + X (A + B_u K)^T + B_w W B_w^T \preceq 0.$$

Recipe I (congruence + change-of-variables):
Apply change-of-variables

$$L := KX$$

to obtain the LMI

$$AX + XA^T + B_u L + L^T B_u^T + B_w W B_w^T \preceq 0$$

As for the cost function, introduce an extra variable Z such that

$$Z \succeq (C_z + D_{zu}K)X(C_z + D_{zu}K)^T$$

and use Schur Complement

$$Z \succeq (C_z + D_{zu}K)X(C_z + D_{zu}K)^T \quad \Leftrightarrow \quad \begin{bmatrix} Z & (C_z + D_{zu}K)X \\ X(C_z + D_{zu}K)^T & X \end{bmatrix} \succeq 0.$$

Applying the change-of-variables $L = KX^{-1}$ this becomes

$$\begin{bmatrix} Z & C_z X + D_{zu} L \\ XC_z^T + L^T D_{zu}^T & X \end{bmatrix} \succeq 0.$$

10.3.2 Summary: LQR revisited (dual form)

The state feedback controller

$$u(t) = Kx(t), \qquad\qquad K = LX^{-1}$$

where $X \in \mathbb{S}^n$, $L \in \mathbb{R}^{m\times n}$ and $Z \in \mathbb{S}^r$ solve the SDP

$$\begin{aligned} \min_{X\in\mathbb{S}^n, L\in\mathbb{R}^{m\times n}, Z\in\mathbb{S}^r} \quad & \mathrm{trace}(Z) \\ \text{s.t.} \quad & AX + XA^T + B_uL + L^TB_u^T + B_wWB_w^T \preceq 0, \\ & \begin{bmatrix} Z & C_zX + D_{zu}L \\ XC_z^T + L^TD_{zu}^T & X \end{bmatrix} \succeq 0, \quad X \succ 0. \end{aligned}$$

stabilizes the closed loop system and minimizes the cost function

$$J := \lim_{t\to\infty} E\left[z(t)^Tz(t)\right]$$

for the LTI system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_uu(t) + B_ww(t), \quad x(0) = 0 \\ z(t) &= C_zx(t) + D_{zu}u(t) \end{aligned}$$

under the assumptions

a) $D_{zu}^TD_{zu} \succ 0$;

b) (A, B_u) stabilizable;

c) $w(t)$ is a Gaussian white noise vector with zero mean and covariance $W \succ 0$.

10.3.3 Summary: Decentralized LQR Control (dual form)

The decentralized state feedback controller

$$u(t) = K_D x(t), \qquad\qquad K_D = L_D X_D^{-1}$$

where $X \in \mathbb{S}^n$, $L \in \mathbb{R}^{m\times n}$ and $Z \in \mathbb{S}^r$ solve the SDP

$$\begin{aligned} \min_{X\in\mathbb{S}^n, L\in\mathbb{R}^{m\times n}, Z\in\mathbb{S}^r} \quad & \mathrm{trace}(Z) \\ \text{s.t.} \quad & AX + XA^T + B_u L + L^T B_u^T + B_w W B_w^T \preceq 0, \\ & \begin{bmatrix} Z & C_z X + D_{zu} L \\ X C_z^T + L^T D_{zu}^T & X \end{bmatrix} \succeq 0, \quad X \succ 0 \\ & X = X_D, \qquad L = L_D, \end{aligned}$$

is a suboptimal solution to the problem of stabilizing the closed loop system and minimizing the cost function

$$J := \lim_{t\to\infty} E\left[z(t)^T z(t)\right] \leq \mathrm{trace}(Z)$$

for the LTI system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_u u(t) + B_w w(t), \quad x(0) = 0 \\ z(t) &= C_z x(t) + D_{zu} u(t) \end{aligned}$$

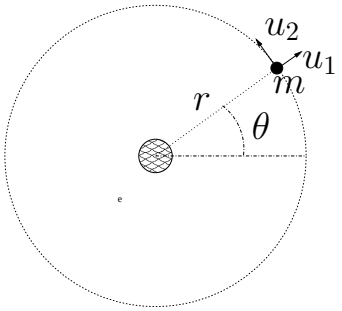
under the assumptions

a) $D_{zu}^T D_{zu} \succ 0$;

b) (A, B_u) stabilizable;

c) $w(t)$ is a Gaussian white noise vector with zero mean and covariance $W \succ 0$.

10.4 Example: satellite in circular orbit



Satellite of mass m with thrust in the radial direction u_1 and in the tangential direction u_2. Continuing...

$$m(\ddot{r} - r\dot{\theta}^2) = u_1 - \frac{km}{r^2} + w_1,$$
$$m(2\dot{r}\dot{\theta} + r\ddot{\theta}) = u_2 + w_2,$$

where w_1 and w_2 are independent white noise disturbances with variances δ_1 and δ_2.
As before, putting in state space and linearizing around

$$\bar{x}(t) = \begin{pmatrix} \bar{r} \\ \bar{\omega}t \\ 0 \\ \bar{\omega} \end{pmatrix}, \qquad \bar{u} = \begin{pmatrix} \bar{u}_1 \\ \bar{u}_2 \end{pmatrix} = 0, \qquad \bar{w} = \begin{pmatrix} \bar{w}_1 \\ \bar{w}_2 \end{pmatrix} = 0, \qquad \bar{\omega} = \sqrt{k/\bar{r}^3}$$

one obtains the linearized system

$$\dot{x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 3\,\bar{\omega}^2 & 0 & 0 & 2\,\bar{r}\bar{\omega} \\ 0 & 0 & -2\,\bar{\omega}/\bar{r} & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1/m & 0 \\ 0 & 1/(m\bar{r}) \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1/m & 0 \\ 0 & 1/(m\bar{r}) \end{bmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.$$

Problem: Given

$$m = 100 \text{ kg}, \qquad \bar{r} = R + 300 \text{ km}, \qquad \bar{k} = GM$$

where $G \approx 6.673 \times 10^{-11}$ N m^2/kg^2 is the universal gravitational constant, and $M \approx 5.98 \times 10^{24}$ kg and $R \approx 6.37 \times 10^3$ km are the mass and radius of the earth. If the variances δ_1 and δ_2 are $0.1N$ find solutions to the LQR control problem where

$$Q = I, \qquad R = \rho I,$$

and u_1 only feedbacks r and $\dot{r}$, u_2 only feedbacks θ and $\dot{\theta}$.

As before apply the similarity transformation

$$\tilde{x} = \begin{pmatrix} r \\ \dot{r} \\ \theta \\ \dot{\theta} \end{pmatrix} = Tx = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} r \\ \theta \\ \dot{r} \\ \dot{\theta} \end{pmatrix}$$

and partition the state

$$\tilde{x} = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix}, \qquad \tilde{x}_1 = \begin{pmatrix} r \\ \dot{r} \end{pmatrix}, \qquad \tilde{x}_2 = \begin{pmatrix} \theta \\ \dot{\theta} \end{pmatrix}$$

so that the decentralized control

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \qquad u_1 = K_1 \tilde{x}_1, \qquad u_2 = K_2 \tilde{x}_2$$

have the desired structure, but now use the dual form.

```
% MAE 280 B - Linear Control Design
% Mauricio de Oliveira
%
% LMI LQR Control
%
m = 100;                % 100 kg
r = 300E3;              % 300 km
R = 6.37E6;             % 6.37 10^3 km
G = 6.673E-11;          % 6.673 N m^2/kg^2
M = 5.98E24;            % 5.98 10^24 kg
k = G * M;              % gravitational force constant
w = sqrt(k/((R+r)^3)); % angular velocity (rad/s)
v = w * (R + r);        % "ground" velocity (m/s)

% linearized system matrices
A = [0 0 1 0; 0 0 0 1; 3*w^2 0 0 2*(r+R)*w; 0 0 -2*w/(r+R) 0];
Bu = [0 0; 0 0; 1/m 0; 0 1/(m*r)];
Bw = [0 0; 0 0; 1/m 0; 0 1/(m*r)];

% noise variances
W = 0.1 * eye(2)
W =
   1.0000e-01            0
            0   1.0000e-01

% scale
T = diag([1 r 1 r])
T =
           1           0           0           0
           0      300000           0           0
           0           0           1           0
```

```
           0           0           0      300000

% similarity transformation
At = T * A / T
At =
            0            0   1.0000e+00            0
            0            0            0   1.0000e+00
   4.0343e-06            0            0   5.1565e-02
            0            0  -1.0432e-04            0
But = T * Bu
But =
            0            0
            0            0
   1.0000e-02            0
            0   1.0000e-02
Bwt = T * Bw
Bwt =
            0            0
            0            0
   1.0000e-02            0
            0   1.0000e-02

% another similarity to change state
T = [1 0 0 0; 0 0 1 0; 0 1 0 0; 0 0 0 1]
T =
     1     0     0     0
     0     0     1     0
     0     1     0     0
     0     0     0     1
Att = T * At / T
Att =
            0   1.0000e+00            0            0
   4.0343e-06            0            0   5.1565e-02
            0            0            0   1.0000e+00
            0  -1.0432e-04            0            0
Butt = T * But
Butt =
            0            0
   1.0000e-02            0
            0            0
            0   1.0000e-02
Bwtt = T * Bwt
Bwtt =
            0            0
   1.0000e-02            0
            0            0
            0   1.0000e-02

% optimal LQR state feedback control (using u1 and u2)
n = size(Att,1);
m = size(Butt,2);

% LQR solution
Cz = [eye(n); zeros(m,n)];
```

```
Dzu = [zeros(n,m); eye(m)];

[K,X,S] = lqr(Att, Butt, Cz' * Cz, Dzu' * Dzu);
Klqr = - K
Klqr =
  -9.8444e-01  -1.3843e+01   1.7795e-01  -2.4919e+00
  -1.7795e-01  -2.4919e+00  -9.8404e-01  -1.4741e+01

% LMI solution with decentralization
% declare variables
X = sdpvar(n,n,'symmetric');
Z = sdpvar(m+n,m+n,'symmetric');
L = sdpvar(m,n);

% declare LMIs
LMI1 = Att*X+X*Att'+Butt*L+L'*Butt'+Bwtt*W*Bwtt';
LMI2 = [Z, Cz*X+Dzu*L; X*Cz'+L'*Dzu', X];

% Structural constraints
Xmask = [zeros(2,2) ones(2,2); ones(2,2) zeros(2,2)]
Xmask =
     0     0     1     1
     0     0     1     1
     1     1     0     0
     1     1     0     0
Lmask = [zeros(1,2) ones(1,2); ones(1,2) zeros(1,2)]
Lmask =
     0     0     1     1
     1     1     0     0

LMI = set(LMI1 < 0) + set(LMI2 > 0) + set(Xmask.*X == 0) + set(Lmask.*L == 0);
options = sdpsettings('solver','sedumi');
solution =solvesdp(LMI,trace(Z),options)
SeDuMi 1.1R3 by AdvOL, 2006 and Jos F. Sturm, 1998-2003.
Alg = 2: xz-corrector, theta = 0.250, beta = 0.500
Put 24 free variables in a quadratic cone
eqs m = 39, order n = 17, dim = 142, blocks = 4
nnz(A) = 95 + 0, nnz(ADA) = 1521, nnz(L) = 780
 it :     b*y       gap    delta  rate   t/tP*  t/tD*   feas cg cg  prec
  0 :            2.29E+00 0.000
  1 :   6.89E-02 4.60E-01 0.000 0.2008 0.9000 0.9000   2.20  1  1  8.0E-01
  2 :   2.73E-04 1.74E-02 0.000 0.0377 0.9900 0.9900   1.36  1  1  2.5E-02
  3 :  -1.31E-05 7.63E-05 0.000 0.0044 0.9990 0.9990   1.02  1  1  1.1E-04
  4 :  -1.69E-04 3.14E-05 0.000 0.4116 0.9000 0.9000   0.14  1  1  1.3E-04
  5 :  -6.40E-04 1.02E-05 0.000 0.3230 0.9000 0.9000  -0.50  1  1  1.2E-04
  6 :  -2.74E-03 2.34E-06 0.000 0.2310 0.9000 0.9000  -0.75  1  1  9.3E-05
  7 :  -1.02E-02 6.18E-07 0.000 0.2634 0.9000 0.9000  -0.90  1  1  8.7E-05
  8 :  -2.00E-02 2.64E-07 0.000 0.4279 0.9000 0.9000  -0.79  1  1  7.1E-05
  9 :  -2.41E-02 9.08E-08 0.000 0.3436 0.9000 0.9000   0.10  1  1  2.7E-05
 10 :  -3.08E-02 2.13E-08 0.000 0.2340 0.9000 0.9000   0.39  1  1  8.4E-06
 11 :  -3.33E-02 1.15E-09 0.000 0.0542 0.9900 0.9900   0.83  1  1  4.9E-07
 12 :  -3.34E-02 1.28E-10 0.000 0.1112 0.9450 0.9450   0.99  1  1  5.5E-08
 13 :  -3.34E-02 2.31E-11 0.003 0.1809 0.9000 0.9000   1.00  1  1  9.9E-09
 14 :  -3.34E-02 4.66E-12 0.000 0.2013 0.9000 0.9000   1.00  2  2  2.0E-09
```

```
 15 :  -3.34E-02 2.81E-13 0.342 0.0603 0.9900 0.9900   1.00  2  2  1.2E-10

iter seconds digits       c*x               b*y
 15      0.1   Inf -3.3440240240e-02 -3.3440235163e-02
|Ax-b| =   1.3e-10, [Ay-c]_+ =   1.6E-11, |x|=  3.5e+03, |y|=  2.1e-02

Detailed timing (sec)
   Pre          IPM          Post
1.000E-02    1.300E-01    1.000E-02
Max-norms: ||b||=1, ||c|| = 1.000000e-05,
Cholesky |add|=0, |skip| = 0, ||L.L|| = 282.571.
solution =
    yalmiptime: 6.2693e-02
    solvertime: 1.4494e-01
          info: 'No problems detected (SeDuMi-1.1)'
       problem: 0
        dimacs: [6.4591e-11 0 0 1.6040e-11 -4.7582e-09 -4.7574e-09]

% Construct K and check closed loop stability and performance
double(L)
ans =
  -4.1530e-03  -5.8787e-04  -5.4486e-14  -1.1347e-12
  -8.1442e-14  -1.1010e-12  -2.9304e-03  -5.6459e-04
double(X)
ans =
   4.1553e-03  -3.1031e-10   3.2566e-15  -3.7495e-13
  -3.1031e-10   4.1513e-05  -1.9902e-13  -9.7943e-12
   3.2566e-15  -1.9902e-13   2.9331e-03  -1.3147e-10
  -3.7495e-13  -9.7943e-12  -1.3147e-10   2.9304e-05
Ld = double(L) - double(L) .* Lmask
Ld =
  -4.1530e-03  -5.8787e-04            0            0
            0            0  -2.9304e-03  -5.6459e-04
Xd = double(X) - double(X) .* Xmask
Xd =
   4.1553e-03  -3.1031e-10            0            0
  -3.1031e-10   4.1513e-05            0            0
            0            0   2.9331e-03  -1.3147e-10
            0            0  -1.3147e-10   2.9304e-05
Klmi = Ld / Xd
Klmi =
  -9.9944e-01  -1.4161e+01            0            0
            0            0  -9.9908e-01  -1.9267e+01
Acl = Att + Butt*Klmi;
eig(Acl)
ans =
  -7.0859e-02 + 7.0494e-02i
  -7.0859e-02 - 7.0494e-02i
  -9.6280e-02 + 2.6849e-02i
  -9.6280e-02 - 2.6849e-02i
Y = lyap(Acl, Bwtt * W * Bwtt')
Y =
   3.6796e-03  -1.2197e-19   3.9952e-04  -4.3182e-10
  -1.2197e-19   3.6761e-05   4.3182e-10   3.9915e-06
```

```
   3.9952e-04   4.3182e-10   2.5973e-03   4.0658e-20
  -4.3182e-10   3.9915e-06   4.0658e-20   2.5949e-05
sqrt(trace(Y))
ans =
   7.9622e-02
sqrt(diag(Y))
ans =
   6.0660e-02
   6.0631e-03
   5.0964e-02
   5.0941e-03

% Try stripping lqr gain
Klqr
Klqr =
  -9.8444e-01  -1.3843e+01   1.7795e-01  -2.4919e+00
  -1.7795e-01  -2.4919e+00  -9.8404e-01  -1.4741e+01
Klqrd = double(Klqr) - double(Klqr) .* Lmask
Klqrd =
  -9.8444e-01  -1.3843e+01            0            0
            0            0  -9.8404e-01  -1.4741e+01
Acl = Att + Butt*Klqrd;
eig(Acl)
ans =
  -6.9537e-02 + 7.0746e-02i
  -6.9537e-02 - 7.0746e-02i
  -7.3382e-02 + 6.6749e-02i
  -7.3382e-02 - 6.6749e-02i
Y = lyap(Acl, Bwtt * W * Bwtt')
Y =
   3.9016e-03   3.3881e-20   6.2032e-04   1.7378e-12
   3.3881e-20   3.8393e-05  -1.7378e-12   6.1042e-06
   6.2032e-04  -1.7378e-12   3.4465e-03   1.2875e-19
   1.7378e-12   6.1042e-06   1.2875e-19   3.3915e-05
sqrt(trace(Y))
ans =
   8.6142e-02
sqrt(diag(Y))
ans =
   6.2463e-02
   6.1962e-03
   5.8707e-02
   5.8236e-03

% LQR solution
Cz = [eye(n); zeros(m,n)];
Dzu = 1e4*[zeros(n,m); eye(m)];

[K,X,S] = lqr(Att, Butt, Cz' * Cz, Dzu' * Dzu);
Klqr = - K
Klqr =
  -1.9615e-05  -6.5712e-03   3.8376e-05  -5.2724e-02
  -1.5950e-04  -5.2724e-02   9.2343e-05  -7.2284e-01
```

```
% LMI solution with decentralization
% declare variables
X = sdpvar(n,n,'symmetric');
Z = sdpvar(m+n,m+n,'symmetric');
L = sdpvar(m,n);

% declare LMIs
LMI1 = Att*X+X*Att'+Butt*L+L'*Butt'+Bwtt*W*Bwtt';
LMI2 = [Z, Cz*X+Dzu*L; X*Cz'+L'*Dzu', X];

% Structural constraints
Xmask = [zeros(2,2) ones(2,2); ones(2,2) zeros(2,2)]
Xmask =
     0     0     1     1
     0     0     1     1
     1     1     0     0
     1     1     0     0
Lmask = [zeros(1,2) ones(1,2); ones(1,2) zeros(1,2)]
Lmask =
     0     0     1     1
     1     1     0     0

LMI = set(LMI1 < 0) + set(LMI2 > 0) + set(Xmask.*X == 0) + set(Lmask.*L == 0);
options = sdpsettings('solver','sedumi');
solution =solvesdp(LMI,trace(Z),options)
SeDuMi 1.1R3 by AdvOL, 2006 and Jos F. Sturm, 1998-2003.
Alg = 2: xz-corrector, theta = 0.250, beta = 0.500
Put 24 free variables in a quadratic cone
eqs m = 39, order n = 17, dim = 142, blocks = 4
nnz(A) = 95 + 0, nnz(ADA) = 1521, nnz(L) = 780
 it :     b*y       gap    delta  rate   t/tP*  t/tD*   feas cg cg  prec
  0 :            2.29E+00 0.000
  1 :   6.89E-02 4.60E-01 0.000 0.2008 0.9000 0.9000   2.20  1  1  8.0E-01
  2 :   2.73E-04 1.74E-02 0.000 0.0377 0.9900 0.9900   1.36  1  1  2.5E-02
  3 :  -1.31E-05 7.63E-05 0.000 0.0044 0.9990 0.9990   1.02  1  1  1.1E-04
  4 :  -1.69E-04 3.14E-05 0.000 0.4117 0.9000 0.9000   0.14  1  1  1.3E-04
  5 :  -6.45E-04 1.01E-05 0.000 0.3228 0.9000 0.9000  -0.50  1  1  1.2E-04
  6 :  -3.19E-03 2.05E-06 0.000 0.2018 0.9000 0.9000  -0.77  1  1  9.4E-05
  7 :  -1.57E-02 4.54E-07 0.000 0.2219 0.9000 0.9000  -0.97  1  1  9.6E-05
  8 :  -6.61E-02 1.17E-07 0.000 0.2575 0.9000 0.9000  -1.05  1  1  1.0E-04
  9 :  -2.73E-01 2.93E-08 0.000 0.2504 0.9000 0.9000  -1.11  1  1  1.0E-04
 10 :  -1.19E+00 7.30E-09 0.000 0.2492 0.9000 0.9000  -1.14  1  1  1.1E-04
 11 :  -5.83E+00 1.53E-09 0.000 0.2091 0.9000 0.9000  -1.12  1  1  1.1E-04
 12 :  -2.89E+01 3.11E-10 0.000 0.2041 0.9000 0.9000  -1.07  1  1  1.1E-04
 13 :  -1.41E+02 6.27E-11 0.000 0.2012 0.9000 0.9000  -1.03  1  1  1.1E-04
 14 :  -6.20E+02 1.34E-11 0.000 0.2142 0.9000 0.9000  -0.99  1  1  1.0E-04
 15 :  -2.20E+03 3.62E-12 0.000 0.2693 0.9000 0.9000  -0.94  4  4  9.8E-05
 16 :  -2.46E+04 3.04E-13 0.000 0.0840 0.9900 0.9900  -0.96  6  6  9.1E-05
 17 :  -4.89E+04 1.31E-13 0.000 0.4319 0.9000 0.9000  -0.77 11 11  7.5E-05
 18 :  -1.39E+05 5.06E-14 0.000 0.3859 0.9000 0.9000  -0.74 11 11  8.1E-05
 19 :  -2.27E+05 2.11E-14 0.000 0.4170 0.9000 0.9000  -0.53  9  9  5.3E-05
 20 :  -3.42E+05 5.36E-15 0.000 0.2540 0.9000 0.9000   0.03 20 20  1.9E-05
 21 :  -3.76E+05 3.91E-16 0.000 0.0729 0.9900 0.9900   0.82 15 15  1.5E-06
 22 :  -3.78E+05 1.32E-16 0.000 0.3377 0.9000 0.9000   0.98 24 22  5.1E-07
```

```
 23 :  -3.79E+05 3.30E-17 0.000 0.2496 0.9000 0.9000   0.99 20 19  1.3E-07
 24 :  -3.79E+05 1.46E-18 0.000 0.0442 0.9900 0.9900   1.00 21 21  5.7E-09
 25 :  -3.79E+05 3.15E-19 0.000 0.2164 0.9000 0.9000   1.00 28 29  1.2E-09
 26 :  -3.79E+05 1.79E-20 0.303 0.0569 0.9900 0.9900   1.00 27 27  7.0E-11

iter seconds digits       c*x               b*y
 26      0.5   Inf -3.7933106979e+05 -3.7933104329e+05
|Ax-b| =   6.8e-07, [Ay-c]_+ =   9.5E-12, |x|=  3.8e+10, |y|=  2.7e+05

Detailed timing (sec)
   Pre          IPM          Post
0.000E+00    5.300E-01    0.000E+00
Max-norms: ||b||=1, ||c|| = 1.000000e-05,
Cholesky |add|=8, |skip| = 0, ||L.L|| = 500000.
solution =
    yalmiptime: 3.4087e-02
    solvertime: 5.3757e-01
          info: 'No problems detected (SeDuMi-1.1)'
       problem: 0
        dimacs: [2.2087e-07 0 0 9.5198e-12 -3.4936e-08 -3.4931e-08]

% Construct K and check closed loop stability and performance
double(L)
ans =
  -1.0051e+00  -1.9109e-03   3.9832e-16  -8.6545e-13
  -1.9023e-15  -8.7300e-13  -1.9149e-02  -6.5467e-04
double(X)
ans =
   1.2270e+03  -5.0097e-05  -2.3332e-17  -1.7193e-15
  -5.0097e-05   5.1018e-03  -1.4309e-16  -8.4199e-13
  -2.3332e-17  -1.4309e-16   1.8884e+02  -6.5158e-07
  -1.7193e-15  -8.4199e-13  -6.5158e-07   1.9150e-04
Ld = double(L) - double(L) .* Lmask
Ld =
  -1.0051e+00  -1.9109e-03            0            0
            0            0  -1.9149e-02  -6.5467e-04
Xd = double(X) - double(X) .* Xmask
Xd =
   1.2270e+03  -5.0097e-05            0            0
  -5.0097e-05   5.1018e-03            0            0
            0            0   1.8884e+02  -6.5158e-07
            0            0  -6.5158e-07   1.9150e-04
Klmi = Ld / Xd
Klmi =
  -8.1924e-04  -3.7456e-01            0            0
            0            0  -1.0142e-04  -3.4185e+00
Acl = Att + Butt*Klmi;
eig(Acl)
ans =
  -3.3978e-02
  -1.9615e-03 + 5.7235e-04i
  -1.9615e-03 - 5.7235e-04i
  -2.9726e-05
Y = lyap(Acl, Bwtt * W * Bwtt')
```

```
Y =
   9.9003e+02  -1.0825e-15   4.8595e+01   4.0278e-03
  -1.0825e-15   3.9089e-03  -4.0278e-03   1.8697e-04
   4.8595e+01  -4.0278e-03   1.4407e+02  -4.3368e-19
   4.0278e-03   1.8697e-04  -4.3368e-19   1.4569e-04
sqrt(trace(Y))
ans =
   3.3677e+01
sqrt(diag(Y))
ans =
   3.1465e+01
   6.2522e-02
   1.2003e+01
   1.2070e-02

% Try stripping lqr gain
Klqr
Klqr =
  -1.9615e-05  -6.5712e-03   3.8376e-05  -5.2724e-02
  -1.5950e-04  -5.2724e-02   9.2343e-05  -7.2284e-01
Klqrd = double(Klqr) - double(Klqr) .* Lmask
Klqrd =
  -1.9615e-05  -6.5712e-03            0            0
            0            0   9.2343e-05  -7.2284e-01
Acl = Att + Butt*Klqrd;
eig(Acl)
ans =
  -6.4410e-03
  -2.6162e-03
   1.6344e-03
   1.2869e-04

diary off
```

10.5 Robust Control

The following discussion is one of many possible approaches to this difficult problem.

10.5.1 Robust Stability

Consider the CTLTI uncertain system

$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0, \qquad x(t) \in \mathbb{R}^n, \qquad A \in \mathcal{A}, \tag{18}$$

where $\mathcal{A}$ is an arbitrary *closed convex set*.

Definition: The CTLTI uncertain system (18) is said to be *robustly stable* if (18) is asymptotically stable for all $A \in \mathcal{A}$.

Robust Stability: Let Ω be the set of stable matrices. If $\mathcal{A} \subseteq \Omega$ then the CTLTI uncertain system (18) is robustly stable.

Proposition: The set of all stable matrices Ω *is not* a convex set.

Proof: The set $\Omega := \{X \in \mathbb{R}^{n\times n} : \max_i \mathrm{Re}\{\lambda_i(X)\} < 0\}$ is a non convex cone.

10.5.2 Polyhedral Uncertainty

Let the set of uncertain systems be characterized by

$$\mathcal{A} := \mathrm{co}\,(A_1, \ldots, A_N) = \left\{ A(\xi) \,:\, A(\xi) = \sum_{i=1}^{N} \xi_i A_i, \quad \forall \xi \in \Xi \right\},$$

$$\Xi := \left\{ \xi \,:\, \xi = (\xi_1, \ldots, \xi_N) \geq 0, \quad \sum_{i=1}^{N} \xi_i = 1 \right\}.$$

Proposition: Any closed convex set $\mathcal{A}$ can be arbitrarily approximated by a convex polyhedron.

Drawback: Often a very large number of vertices is required.

10.5.3 Quadratic Stability

The CTLTI uncertain system (18) where $\mathcal{A} = \mathrm{co}\,(A_1, \ldots, A_N)$ is *Quadratically Stable* if there exists $P \in \mathbb{S}^n$ such that the LMI

$$P \succ 0, \qquad A_i^T P + P A_i \prec 0, \qquad \text{for all } i = 1, \ldots, N,$$

is feasible.

Proposition: If the CTLTI uncertain system (18) where $\mathcal{A} = \mathrm{co}\,(A_1, \ldots, A_N)$ is quadratically stable then it is also robustly stable.

Proof: For any $\tilde{A} \in \mathcal{A}$ there exists a $\tilde{\xi} \in \Xi$ such that $\tilde{A} = \sum_{i=1}^N \tilde{\xi}_i A_i$. Since $\tilde{\xi} \in \Xi \Rightarrow \tilde{\xi}_i \geq 0$, multiply each inequality in the QS condition by $\tilde{\xi}_i$ to obtain

$$P \succ 0, \qquad \left(\tilde{\xi}_i A_i^T\right) P + P \left(\tilde{\xi}_i A_i\right) \prec 0, \qquad \text{for all } i = 1, \ldots, N.$$

The sum of all the above inequalities provides

$$P \succ 0, \qquad \sum_{i=1}^N \left(\tilde{\xi}_i A_i^T\right) P + P \sum_{i=1}^N \left(\tilde{\xi}_i A_i\right) = \tilde{A}^T P + P \tilde{A} \prec 0$$

which implies that $\tilde{A}$ is stable. The proof is completed by noticing that the above holds for any $A \in \mathcal{A}$.

Remarks:

a) Robust stability requires stability to be checked for all $\xi \in \Xi$. This is equivalent to solve an infinite number of SDP feasibility problems (Lyapunov inequalities) on $P(\xi) \succ 0$. The resulting $V(x, \xi) = x^T P(\xi)\, x$ is called a *parameter-dependent Lyapunov function*.

b) *Quadratic stability* requires stability to be checked at the N vertices of $\mathcal{A}$. This is equivalent to solve N SDP feasibility problems (Lyapunov inequalities) on $P \succ 0$. The resulting $V(x, \xi) = V(x) = x^T P x$ does not depend on ξ.

c) Quadratic stability $\not\Rightarrow$ robust stability.

d) Quadratic stability is broader. Consider the time-varying linear system

$$\dot{x}(t) = A(\xi(t))x(t), \qquad x(0) = x_0, \qquad \xi(t) \in \Xi \text{ for all } t \geq 0.$$

If $\mathcal{A} = \mathrm{co}\,(A_1, \ldots, A_N)$ is quadratically stable then the origin is an asymptotically stable equilibrium. This is because $V(x) = x^T P x$ is a Lyapunov function and

$$\begin{aligned} \dot{V}(x(t)) &= 2x(t)^T P \dot{x}(t) \\ &= x(t)^T [A(\xi(t))^T P + P A(\xi(t))] x(t) < 0 \end{aligned}$$

for all $x(t) \neq 0$. Note that $\dot{V}(x(t))$ does not depend on the $\dot{\xi}(t)$ so that ξ can vary *arbitrarily fast*!

10.5.4 Example

Find $\underline{\theta} < \theta < \overline{\theta}$ such that the CTLTI system

$$H(s) = \frac{N_H(s)}{D_H(s)} = \frac{2s^3 - s^2 + 1}{s^4 + s^3 + 25s^2 + 3(4+\theta)s + 3(4+\theta)}$$

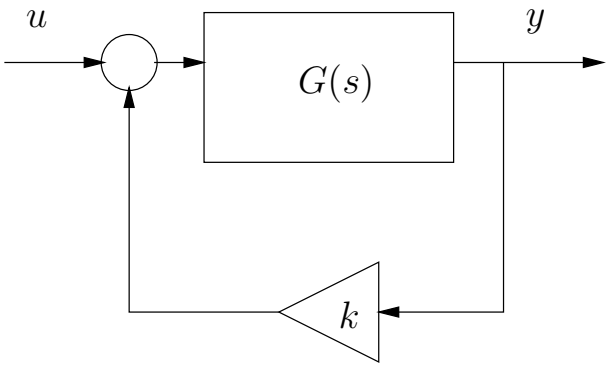
is robustly stable. In state-space

$$\dot{x} = A(\theta)x, \quad A(\theta) \in \operatorname{co}\left(A(\underline{\theta}), A(\overline{\theta})\right),$$

$$A(\theta) := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -3(4+\theta) & -3(4+\theta) & -25 & -1 \end{bmatrix}$$

Results:

Stability	$\underline{\theta}$	$\overline{\theta}$
Robust	-4	4
Quadratic	-1.8	2.2



A feedback control connection:

$$\frac{Y(s)}{U(s)} = \frac{G(s)}{1 + kG(s)} = \frac{N_G(s)}{D_G(s) + kN_G(s)}$$

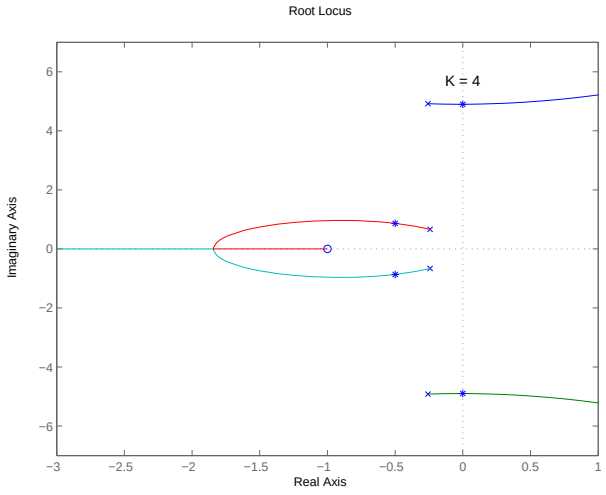
Hence, an appropriate $G(s)$ for stability analysis can be computed from $D_G(s) + kN_G(s) = D_H(s)$, that is

$$\underbrace{(s^4 + s^3 + 25s^2 + 12s + 12)}_{D_G(s)} + k\underbrace{3(s+1)}_{N_G(s)} = \underbrace{s^4 + s^3 + 25s^2 + 3(4+k)s + 3(4+k)}_{D_H(s)}.$$

Set

$$G(s) = \frac{N_G(s)}{D_G(s)} = \frac{3(s+1)}{s^4 + s^3 + 25s^2 + 12s + 12}$$

and, from root locus plot, it can be shown that the CTLTI system is (robustly) stable for $-4 < \theta < 4$. Alternatively, one could also use the Routh-Hurwitz criterion directly on $D_H(s)$.



10.5.5 Robust LQR Control

Problem: Given the uncertain LTI system

$$\begin{aligned}\dot{x}(t) &= A(\xi)x(t) + B_u(\xi)u(t) + B_w w(t), \qquad x(0) = 0, \qquad x \in \mathbb{R}^n,\\ z(t) &= C_z x(t) + D_{zu}u(t).\end{aligned}$$

where $\begin{bmatrix} A(\xi) & B_u(\xi)\end{bmatrix} \in \mathcal{M}$,

$$\mathcal{M} := \text{co}\,(M_1, \ldots, M_N) = \left\{ M(\xi) \,:\, M(\xi) = \sum_{i=1}^{N} \xi_i M_i, \quad \forall \xi \in \Xi \right\},$$

$$\Xi := \left\{ \xi \,:\, \xi = (\xi_1, \ldots, \xi_N) \geq 0, \quad \sum_{i=1}^{N} \xi_i = 1 \right\}.$$

compute a state feedback controller

$$u(t) = K\, x(t),$$

that stabilizes the closed loop system and minimizes

$$J := \lim_{t\to\infty} E\left[z(t)^T z(t)\right]$$

Assumptions:

a) $D_{zu}^T D_{zu} \succ 0$;

b) $w(t)$ is a Gaussian white noise vector with zero mean and covariance $W \succ 0$.

Solution: We do not know of a complete solution to this problem. Here we will be happy with a sufficient design (conservative) using quadratic stability.

Using the congruence+change-of-variables the LQR problem can be recast as the SDP

$$\begin{aligned} \min_{X,L,Z} \quad & \mathrm{trace}(ZW) \\ \text{s.t.} \quad & \begin{bmatrix} AX + XA^T + B_u L + L^T B_u^T & XC_z^T + L^T D_{zu}^T \\ C_z X + D_{zu} L & -I \end{bmatrix} \preceq 0, \\ & \begin{bmatrix} Z & B_w^T \\ B_w & X \end{bmatrix} \succeq 0, \quad X \succ 0. \end{aligned}$$

where $K = LX^{-1}$. A natural extension using quadratic stability is that

$$\begin{aligned} \min_{X,L,Z} \quad & \mathrm{trace}(ZW) \\ \text{s.t.} \quad & \begin{bmatrix} A_i X + XA_i^T + B_{ui} L + L^T B_{ui}^T & XC_z^T + L^T D_{zu}^T \\ C_z X + D_{zu} L & -I \end{bmatrix} \preceq 0, \;\; i = 1, \ldots, N, \\ & \begin{bmatrix} Z & B_w^T \\ B_w & X \end{bmatrix} \succeq 0, \quad X \succ 0. \end{aligned}$$

Multiplying each inequality by $\xi_i \geq 0$ and summing we obtain

$$\begin{bmatrix} A(\xi)X + XA(\xi)^T + B_u(\xi) L + L^T B_u(\xi)^T & XC_z^T + L^T D_{zu}^T \\ C_z X + D_{zu} L & -I \end{bmatrix} \preceq 0$$

for all $\xi \in \Xi$. Using the Comparison Lemma $X \succeq P(\xi)$ where $P(\xi)$ is the parameter dependent observability Gramian computed with the controller $K = LX^{-1}$ so that

$$\mathrm{trace}(ZW) \geq J = \lim_{t \to \infty} E\left[z(t)^T z(t)\right]$$

The dual form follows the same pattern.

10.5.6 Summary: Robust LQR Control

The state feedback controller

$$u(t) = Kx(t), \qquad\qquad K = LX^{-1}$$

where $X \in \mathbb{S}^n$, $L \in \mathbb{R}^{m\times n}$ and $Z \in \mathbb{S}^r$ solve the SDP

$$\begin{aligned}
\min_{X,L,Z} \quad & \text{trace}(ZW) \\
\text{s.t.} \quad & \begin{bmatrix} A_iX + XA_i^T + B_{u_i}L + L^TB_{u_i}^T & XC_z^T + L^TD_{zu}^T \\ C_zX + D_{zu}L & -I \end{bmatrix} \preceq 0, \;\; i = 1,\ldots,N, \\
& \begin{bmatrix} Z & B_w^T \\ B_w & X \end{bmatrix} \succeq 0, \quad X \succ 0,
\end{aligned}$$

is a suboptimal solution to the problem of stabilizing the closed loop system and minimizing the cost function

$$J := \lim_{t\to\infty} E\left[z(t)^Tz(t)\right] \leq \text{trace}(ZW)$$

for the uncertain LTI system

$$\begin{aligned}
\dot{x}(t) &= A(\xi)x(t) + B_u(\xi)u(t) + B_ww(t), \quad x(0) = 0 \\
z(t) &= C_zx(t) + D_{zu}u(t)
\end{aligned}$$

where $\begin{bmatrix} A(\xi) & B_u(\xi) \end{bmatrix} \in \mathcal{M}$,

$$\mathcal{M} := \text{co}\,(M_1,\ldots,M_N) = \left\{ M(\xi) \,:\, M(\xi) = \sum_{i=1}^N \xi_iM_i, \quad \forall \xi \in \Xi \right\},$$

$$\Xi := \left\{ \xi \,:\, \xi = (\xi_1,\ldots,\xi_N) \geq 0, \quad \sum_{i=1}^N \xi_i = 1 \right\},$$

under the assumptions

a) $D_{zu}^TD_{zu} \succ 0$;

b) $w(t)$ is a Gaussian white noise vector with zero mean and covariance $W \succ 0$.

10.5.7 Summary: Robust LQR Control (dual form)

The state feedback controller

$$u(t) = Kx(t), \qquad\qquad K = LX^{-1}$$

where $X \in \mathbb{S}^n$, $L \in \mathbb{R}^{m\times n}$ and $Z \in \mathbb{S}^r$ solve the SDP

$$\begin{aligned}
\min_{X,L,Z} \quad & \text{trace}(Z) \\
\text{s.t.} \quad & A_iX + XA_i^T + B_{ui}L + L^TB_{ui}^T + B_wWB_w^T \preceq 0, \qquad i = 1,\ldots,N, \\
& \begin{bmatrix} Z & C_zX + D_{zu}L \\ XC_z^T + L^TD_{zu}^T & X \end{bmatrix} \succeq 0, \quad X \succ 0,
\end{aligned}$$

is a suboptimal solution to the problem of stabilizing the closed loop system and minimizing the cost function

$$J := \lim_{t\to\infty} E\left[z(t)^Tz(t)\right] \leq \text{trace}(Z)$$

for the uncertain LTI system

$$\begin{aligned}
\dot{x}(t) &= A(\xi)x(t) + B_u(\xi)u(t) + B_ww(t), \quad x(0) = 0 \\
z(t) &= C_zx(t) + D_{zu}u(t)
\end{aligned}$$

where $\begin{bmatrix} A(\xi) & B_u(\xi) \end{bmatrix} \in \mathcal{M}$,

$$\mathcal{M} := \text{co}\,(M_1,\ldots,M_N) = \left\{ M(\xi) \,:\, M(\xi) = \sum_{i=1}^N \xi_iM_i, \quad \forall \xi \in \Xi \right\},$$

$$\Xi := \left\{ \xi \,:\, \xi = (\xi_1,\ldots,\xi_N) \geq 0, \quad \sum_{i=1}^N \xi_i = 1 \right\},$$

under the assumptions

a) $D_{zu}^TD_{zu} \succ 0$;

b) $w(t)$ is a Gaussian white noise vector with zero mean and covariance $W \succ 0$.