1 Controllability and Observability

1.1 Linear Time-Invariant (LTI) Systems

State-space:

$$\begin{aligned} \dot{x} &= Ax + Bu, \quad x(0) = x_0, \\ y &= Cx + Du. \end{aligned}$$

Dimensions:

$$x \in \mathbb{R}^n, \qquad u \in \mathbb{R}^m, \qquad y \in \mathbb{R}^p.$$

Notation

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

Transfer function:

$$H(s) = C(sI - A)^{-1}B + D$$

Note that $H(s)$ is always proper!

Similarity transformation:

$$\left[\begin{array}{c|c} T^{-1}AT & T^{-1}B \\ \hline CT & D \end{array}\right] \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

Similarity does not change transfer function

$$H(s) = CT(sI - T^{-1}AT)^{-1}T^{-1}B + D = C(sI - A)^{-1}B + D$$

System response:

$$Y(s) = \underbrace{H(s)U(s)}_{\text{Input}} + \underbrace{C(sI - A)^{-1}x(0)}_{\text{Initial conditions}}$$

MIMO comes for free!

1.2 Concepts from MAE 280 A

Controllability Matrix:

$$\mathcal{C}(A,B) = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}.$$

Controllability Gramian:

$$X(t) = \int_0^t e^{A\xi} B B^T e^{A^T \xi} d\xi.$$

Observability Matrix:

$$\mathcal{O}(A,C) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}.$$

Observability Gramian:

$$Y(t) = \int_0^t e^{A^T \xi} C^T C e^{A\xi} d\xi.$$

1.3 Controllability

Problem: Given $x(0) = 0$ and *any* $\bar{x}$, can one compute $u(t)$ such that $x(\bar{t}) = \bar{x}$ for some $\bar{t} > 0$?

Theorem: The following are equivalent

a) The pair (A, B) is controllable;

b) The Controllability Matrix $\mathcal{C}(A, B)$ has full-row rank;

c) There exists no $z \neq 0$ such that $z^* A = \lambda z^*, \quad z^* B = 0$;

d) The Controllability Gramian $X(t)$ is positive definite for some $t \geq 0$.

1.4 Observability

Problem: Given $y(t)$ over $t \in [0, \bar{t}]$ with $\bar{t} > 0$ can one compute $x(t)$ for all $t \in [0, \bar{t}]$?

Theorem: The following are equivalent

a) The pair (A, C) is observable;

b) The Observability Matrix $\mathcal{O}(A, C)$ has full-column rank;

c) There exists no $x \neq 0$ such that $Ax = \lambda x, \quad Cx = 0$;

d) The Observability Gramian $Y = Y(t)$ is positive definite for some $t \geq 0$.

1.5 Things you should already know

1. Why a) and b) are equivalent.

2. Why can we stop $\mathcal{C}(A, B)$ at $A^{n-1}B$ and $\mathcal{O}(A, C)$ at CA^{n-1}?

3. Kalman canonical forms. E.g. if (A, C) is not observable then

$$\left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} A_o & 0 & B_o \\ A_{\bar{o}o} & A_{\bar{o}} & B_{\bar{o}} \\ \hline C_o & 0 & 0 \end{array}\right]$$

where $A_o \in \mathbb{R}^{r \times r}$ and (A_o, C_o) is observable.

1.6 The Popov-Belevitch-Hautus Test

Theorem: The pair (A, C) is observable if and only if there exists no $x \neq 0$ such that

$$Ax = \lambda x, \quad Cx = 0. \tag{1}$$

Proof:

Sufficiency: Assume there exists $x \neq 0$ such that (1) holds. Then

$$\begin{aligned} CAx &= \lambda Cx = 0, \\ CA^2x &= \lambda CAx = 0, \\ &\vdots \\ CA^{n-1}x &= \lambda CA^{n-2}x = 0 \end{aligned}$$

so that

$$\mathcal{O}(A, C)x = 0,$$

which implies that the pair (A, C) is not observable.

Necessity: Assume that (A, C) is not observable. Then transform it into the equivalent non observable realization where

$$\bar{A} = \begin{bmatrix} A_o & 0 \\ A_{\bar{o}o} & A_{\bar{o}} \end{bmatrix}, \qquad \bar{C} = \begin{bmatrix} C_o & 0 \end{bmatrix}.$$

Chose $x \neq 0$ such that

$$A_{\bar{o}}x = \lambda x.$$

Then

$$\begin{bmatrix} A_o & 0 \\ A_{\bar{o}o} & A_{\bar{o}} \end{bmatrix} \begin{pmatrix} 0 \\ x \end{pmatrix} = \lambda \begin{pmatrix} 0 \\ x \end{pmatrix}, \quad \begin{bmatrix} C_o & 0 \end{bmatrix} \begin{pmatrix} 0 \\ x \end{pmatrix} = 0.$$

1.7 Controllability Gramian

Problem: Given $x(0) = 0$ and *any* $\bar{x}$, compute $u(t)$ such that $x(\bar{t}) = \bar{x}$ for some $\bar{t} > 0$.

Solution: We know that

$$\bar{x} = x(\bar{t}) = \int_0^{\bar{t}} e^{A(\bar{t}-\tau)} B u(\tau) d\tau.$$

If we limit our search to controls u of the form

$$u(t) = B^T e^{A^T(\bar{t}-t)} \bar{z}$$

we have

$$\begin{aligned}
\bar{x} &= \int_0^{\bar{t}} e^{A(\bar{t}-\tau)} B B^T e^{A^T(\bar{t}-\tau)} \bar{z} d\tau, \\
&= \left(\int_0^{\bar{t}} e^{A(\bar{t}-\tau)} B B^T e^{A^T(\bar{t}-\tau)} d\tau \right) \bar{z}, \qquad \xi = \bar{t} - \tau \\
&= \left(\int_0^{\bar{t}} e^{A\xi} B B^T e^{A^T\xi} d\xi \right) \bar{z},
\end{aligned}$$

and

$$\bar{z} = \left(\int_0^{\bar{t}} e^{A\xi} B B^T e^{A^T\xi} d\xi \right)^{-1} \bar{x},$$

$$\Rightarrow \qquad u(t) = B^T e^{A^T(\bar{t}-t)} \left(\int_0^{\bar{t}} e^{A\xi} B B^T e^{A^T\xi} d\xi \right)^{-1} \bar{x}$$

The symmetric matrix

$$X(t) := \int_0^{t} e^{A\xi} B B^T e^{A^T\xi} d\xi$$

is the Controllability Gramian.

1.8 Stabilizability

Problem: Given *any* $x(0) = \bar{x}$ can one compute $u(t)$ such that $x(\bar{t}) = 0$ for some $\bar{t} > 0$?

Theorem: The following are equivalent

a) The pair (A, B) is stabilizable;

b) There exists no $z \neq 0$ and λ such that $z^* A = \lambda z^*, \quad z^* B = 0$ with $\lambda + \lambda^* \geq 0$.

1.9 Detectability

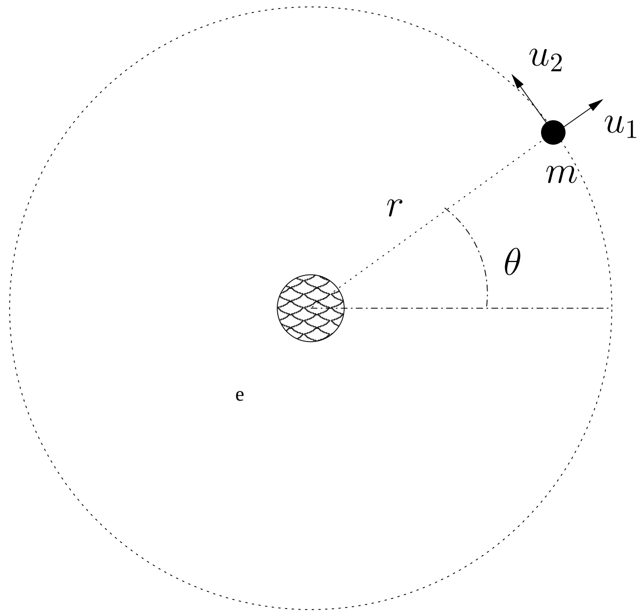
Problem: Given $y(t)$ over $t \in [0, \bar{t}]$ with $\bar{t} > 0$ can one compute $x(\bar{t})$?

Theorem: The following are equivalent

a) The pair (A, C) is detectable;

b) There exists no $x \neq 0$ and λ such that $Ax = \lambda x, \quad Cx = 0$ with $\lambda + \lambda^* \geq 0$.

1.10 Example: satellite in circular orbit



Satellite of mass m with thrust in the radial direction u_1 and in the tangential direction u_2. From Skelton, DSC, p. 101.

Newton's law

$$m\ddot{\vec{r}} = \vec{u}_1 + \vec{u}_2 + \vec{f}_g,$$

where $\vec{f}_g$ is the gravitational force

$$\vec{f}_g = -\frac{km}{r^2}\frac{\vec{r}}{r}.$$

Using cylindrical coordinates

$$\vec{e}_1 = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \qquad \vec{e}_2 = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix},$$

we have

$$\vec{r} = r\vec{e}_1, \qquad \vec{u}_1 = u_1\vec{e}_1, \qquad \vec{u}_2 = u_2\vec{e}_2, \qquad \vec{f}_g = -\frac{km}{r^2}\vec{e}_1.$$

We need to compute

$$\ddot{\vec{r}} = \frac{d^2}{dt^2}(r\vec{e}_1) = \frac{d}{dt}(\dot{r}\vec{e}_1 + r\dot{\vec{e}}_1) = \ddot{r}\vec{e}_1 + 2\dot{r}\dot{\vec{e}}_1 + r\ddot{\vec{e}}_1,$$

where

$$\begin{aligned}
\dot{\vec{e}}_1 &= \dot{\theta}\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} = \dot{\theta}\vec{e}_2, \\
\ddot{\vec{e}}_1 &= \ddot{\theta}\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} + \dot{\theta}^2\begin{pmatrix} -\cos\theta \\ -\sin\theta \end{pmatrix} = \ddot{\theta}\vec{e}_2 - \dot{\theta}^2\vec{e}_1.
\end{aligned}$$

That is

$$\ddot{\vec{r}} = (\ddot{r} - r\dot{\theta}^2)\vec{e}_1 + (2\dot{r}\dot{\theta} + r\ddot{\theta})\vec{e}_2,$$

so that Newton's law can be rewritten as

$$m(\ddot{r} - r\dot{\theta}^2)\vec{e}_1 + m(2\dot{r}\dot{\theta} + r\ddot{\theta})\vec{e}_2 = u_1\vec{e}_1 + u_2\vec{e}_2 - \frac{km}{r^2}\vec{e}_1,$$

or, equivalently, as the two scalar differential equations

$$\begin{aligned}
m(\ddot{r} - r\dot{\theta}^2) &= u_1 - \frac{km}{r^2}, \\
m(2\dot{r}\dot{\theta} + r\ddot{\theta}) &= u_2.
\end{aligned}$$

In state space

$$x = \begin{pmatrix} r \\ \theta \\ \dot{r} \\ \dot{\theta} \end{pmatrix}, \quad \dot{x} = \begin{pmatrix} \dot{r} \\ \dot{\theta} \\ \ddot{r} \\ \ddot{\theta} \end{pmatrix} = \begin{pmatrix} x_3 \\ x_4 \\ x_1x_4^2 - k/x_1^2 \\ -2x_3x_4/x_1 \end{pmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1/m & 0 \\ 0 & 1/(mx_1) \end{bmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

This is a *nonlinear system* and we look for equilibrium ($\ddot{r} = \ddot{\theta} = 0$) when $u_1 = u_2 = 0$. This can be stated as

$$x_1x_4^2 - k/x_1^2 = 0, \qquad\qquad -2x_3x_4/x_1 = 0.$$

The second condition implies $x_3 = \dot{r}$ and/or $x_4 = \dot{\theta}$ must be zero. We choose $x_3 = \dot{r} = 0$ which implies $x_1 = r = \bar{r}$ constant and

$$x_4 = \dot{\theta} = \sqrt{\frac{k}{x_1^3}} = \sqrt{\frac{k}{\bar{r}^3}} = \bar{\omega} \quad \Rightarrow \quad k = \bar{r}^3\bar{\omega}^2.$$

Note also that $x_2 = \theta = \bar{\omega}t$.

This nonlinear system is in the form

$$\dot{x} = f(x,t) + g(x)u.$$

We will linearize $f(x,t)$ and $g(x)u$ around the equilibrium point $(\bar{x}, \bar{u})$ to obtain the linearized system

$$\dot{x} = (\nabla f_x)^T [x(t) - \bar{x}(t)] + (\nabla g_x)^T [x(t) - \bar{x}(t)]\bar{u} + g(\bar{x})u.$$

For this problem

$$\bar{x}(t) = \begin{pmatrix} \bar{r} \\ \bar{\omega} t \\ 0 \\ \bar{\omega} \end{pmatrix}, \quad \bar{u} = 0, \quad f(x,t) = \begin{pmatrix} x_3 \\ x_4 \\ x_1 x_4^2 - k/x_1^2 \\ -2x_3 x_4/x_1 \end{pmatrix}, \quad g(x) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1/m & 0 \\ 0 & 1/(m x_1) \end{bmatrix},$$

and

$$(\nabla f_x)^T = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2\,k/\bar{r}^3 + \bar{\omega}^2 & 0 & 0 & 2\,\bar{r}\bar{\omega} \\ 0 & 0 & -2\,\bar{\omega}/\bar{r} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 3\,\bar{\omega}^2 & 0 & 0 & 2\,\bar{r}\bar{\omega} \\ 0 & 0 & -2\,\bar{\omega}/\bar{r} & 0 \end{bmatrix}.$$

This produces the linearized system

$$\dot{x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 3\,\bar{\omega}^2 & 0 & 0 & 2\,\bar{r}\bar{\omega} \\ 0 & 0 & -2\,\bar{\omega}/\bar{r} & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1/m & 0 \\ 0 & 1/(m\bar{r}) \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

If we looking at the satellite (from the earth) we can say that we can observe r and $\dot{\theta}$ (how?), that is

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x.$$

Questions:

1) Can we estimate the state of the satellite is by measuring only r?
2) Can we estimate the state of the satellite is by measuring only $\dot{\theta}$?
3) Can we estimate the state of the satellite by measuring r and $\dot{\theta}$?
4) Can the system be controlled to remain in circular orbit using radial thrusting (u_1) alone?
5) Can the system be controlled using tangential thrusting (u_2) alone?

Question: Can we estimate the state of the satellite by measuring only r?

Answer: Is the system observable when $C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$? Compute the observability matrix

$$\mathcal{O}(A, C) = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix},$$
$$= \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ \hline 3\bar{\omega}^2 & 0 & 0 & 2\bar{r}\bar{\omega} \\ \hline 0 & 0 & -\bar{\omega}^2 & 0 \end{array}\right]$$

Physical interpretation: measuring r does not give any information on θ or $\bar{\theta}$!

Note that if we that know the satellite is in equilibrium and "measure" k then

$$\dot{\theta} = \bar{\omega} = \sqrt{\frac{k}{\bar{r}^3}}.$$

But we still do not know θ since

$$\theta(t) = \theta(0) + \omega t,$$

and we do not know $\theta(0)$!

Question: Can we estimate the state of the satellite by observing $\dot{\theta}$ only?

Answer: Is the system observable when $C = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$? Compute the observability matrix

$$\mathcal{O}(A, C) = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix},$$
$$= \left[\begin{array}{cccc} 0 & 0 & 0 & 1 \\ \hline 0 & 0 & -2\bar{\omega}/\bar{r} & 0 \\ \hline -6\bar{\omega}^3/\bar{r} & 0 & 0 & -4\bar{\omega}^2 \\ \hline 0 & 0 & 2\bar{\omega}^3/\bar{r} & 0 \end{array}\right]$$

Physical interpretation: again, if we try to reconstruct θ from $\dot{\theta}$ we still need to know θ at some $\bar{t}$! From that point on

$$\theta = \theta(\bar{t}) + \int_{\bar{t}}^{t} \dot{\theta}(\tau) d\tau.$$

Question: Can we estimate the state of the satellite by measuring r and $\dot{\theta}$?

Answer: Is the system observable when $C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$? Compute the observability matrix

$$
\begin{aligned}
\mathcal{O}(A, C) &= \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix}, \\
&= \left[\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
\hline
0 & 0 & 1 & 0 \\
0 & 0 & -2\bar{\omega}/\bar{r} & 0 \\
\hline
3\bar{\omega}^2 & 0 & 0 & 2\bar{r}\bar{\omega} \\
-6\bar{\omega}^3/\bar{r} & 0 & 0 & -4\bar{\omega}^2 \\
\hline
0 & 0 & -\bar{\omega}^2 & 0 \\
0 & 0 & 2\bar{\omega}^3/\bar{r} & 0
\end{array}\right]
\end{aligned}
$$

Physical interpretation: can we estimate θ at all?

Question: Can the system be controlled to remain in circular orbit using radial thrusting (u_1) alone?

Answer: Is the system controllable when $B = \begin{bmatrix} 0 \\ 0 \\ 1/m \\ 0 \end{bmatrix}$? Compute the controllability matrix

$$
\begin{aligned}
\mathcal{O}(A, C) &= \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix}, \\
&= \frac{1}{m} \left[\begin{array}{c|c|c|c} 0 & 1 & 0 & -\bar{\omega}^2 \\ 0 & 0 & -2\bar{\omega}/\bar{r} & 0 \\ 1 & 0 & -\bar{\omega}^2 & 0 \\ 0 & -2\bar{\omega}/\bar{r} & 0 & 2\bar{\omega}^3/\bar{r} \end{array} \right]
\end{aligned}
$$

Note that

$$
-\bar{\omega}^2 \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2\bar{\omega}/\bar{r} \end{pmatrix} = \begin{pmatrix} -\bar{\omega}^2 \\ 0 \\ 0 \\ 2\bar{\omega}^3/\bar{r} \end{pmatrix}
$$

which implies that the system is not controllable from u_1!

Physical interpretation: there must be a change in the angular velocity $\dot{\theta}$ if one changes the radius!

Question: Can the system be controlled using tangential thrusting (u_2) alone?

Answer: Is the system controllable when $B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/(m\bar{r}) \end{bmatrix}$? Compute the controllability matrix

$$\begin{aligned}
\mathcal{O}(A, C) &= \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix}, \\
&= \frac{1}{m\bar{r}} \left[\begin{array}{c|c|c|c} 0 & 0 & 2\bar{r}\bar{\omega} & 0 \\ 0 & 1 & 0 & -4\bar{\omega}^2 \\ 0 & 2\bar{r}\bar{\omega} & 0 & -2\bar{r}\bar{\omega}^3 \\ 1 & 0 & -4\bar{\omega}^2 & 0 \end{array} \right]
\end{aligned}$$

The above matrix has full rank, so the system is controllable from u_2!

Physical interpretation: we can change the radius by changing the angular velocity!

1.11 More on Gramians

Theorem: The Controllability Gramian

$$X(t) = \int_0^t e^{A\xi} B B^T e^{A^T \xi} d\xi,$$

is the solution to the differential equation

$$\frac{d}{dt} X(t) = A X(t) + X(t) A^T + B B^T.$$

If $X = \lim_{t \to \infty} X(t)$ exists then

$$A X + X A^T + B B^T = 0.$$

Theorem: The Observability Gramian

$$Y(t) = \int_0^t e^{A^T \xi} C^T C e^{A\xi} d\xi,$$

is the solution to the differential equation

$$\frac{d}{dt} Y(t) = A^T Y(t) + Y(t) A + C^T C.$$

If $Y = \lim_{t \to \infty} X(t)$ exists then

$$A^T Y + Y A + C^T C = 0.$$

Proof (Controllability): For the first part, compute

$$
\begin{aligned}
\frac{d}{dt}X(t) &= \frac{d}{dt}\int_0^t e^{A\xi}BB^T e^{A^T\xi}d\xi = \frac{d}{dt}\int_0^t e^{A(t-\tau)}BB^T e^{A^T(t-\tau)}d\tau, \\
&= \int_0^t \frac{d}{dt}e^{A(t-\tau)}BB^T e^{A^T(t-\tau)} + e^{A(t-\tau)}BB^T e^{A^T(t-\tau)}\Big|_{\tau=t}, \\
&= A\left(\int_0^t e^{A(t-\tau)}BB^T e^{A^T(t-\tau)}d\tau\right) \\
&\qquad\qquad + \left(\int_0^t e^{A(t-\tau)}BB^T e^{A^T(t-\tau)}d\tau\right)A^T + BB^T, \\
&= AX(t) + X(t)A^T + BB^T.
\end{aligned}
$$

For the second part, use the fact that $X(t)$ is smooth and therefore

$$
\lim_{t\to\infty} X(t) = X \quad \Rightarrow \quad \lim_{t\to\infty}\frac{d}{dt}X(t) = 0.
$$

Lemma: Consider the Lyapunov Equation

$$A^T X + XA + C^T C = 0$$

where $A \in \mathbb{C}^{n\times n}$ and $C \in \mathbb{C}^{m\times n}$.

1. A solution $X \in \mathbb{C}^{n\times n}$ exists and is unique if and only if $\lambda_j(A) + \lambda_i^*(A) \neq 0$ for all $i, j = 1, \ldots, n$. Furthermore X is symmetric.
2. If A is Hurwitz then X is positive semidefinite.
3. If (A, C) is detectable and X is positive semidefinite then A is Hurwitz.
4. If (A, C) is observable and A is Hurwitz then X is positive definite.

Proof:
Item 1. The Lyapunov Equation is a linear equation and it has a unique solution if and only if the homogeneous equation associated with the Lyapunov equation admits only the trivial solution. Assume it does not, that is, there $\bar{X} \neq 0$ such that

$$A^T \bar{X} + \bar{X} A = 0$$

Then, multiplication of the above on the right by $x_i^* \neq 0$, the ith eigenvector of A and on the right by $x_j^* \neq 0$ yields

$$0 = x_i^* A^T \bar{X} x_j + x_i^* \bar{X} A x_j = [\lambda_j(A) + \lambda_i^*(A)]\, x_i^* \bar{X} x_j.$$

Since $\lambda_i(A) + \lambda_j(B) \neq 0$ by hypothesis we must have $x_i^* \bar{X} x_j = 0$ for all i, j. One can show that this indeed implies $\bar{X} = 0$, establishing a contradiction. That X is symmetric follows from uniqueness since

$$\begin{aligned} 0 &= (A^T X + XA + C^T C)^T - (A^T X + XA + C^T C) \\ &= A^T (X^T - X) + (X^T - X) A \end{aligned}$$

so that $X^T - X = 0$.

Item 2. If A is Hurwitz then $\lim_{t\to\infty} e^{At} = 0$. But

$$X = \int_0^\infty e^{A^T t} C^T C e^{At}\, dt \succeq 0$$

and

$$A^T X + XA = \lim_{t\to\infty} \int_0^\infty \frac{d}{dt} e^{A^T t} C^T C e^{At}\, dt = e^{A^T t} C^T C e^{At}\Big|_0^\infty = -C^T C.$$

Item 3. Assume (A, C) is detectable, $X \succeq 0$ and that A is not Hurwitz. Then there exists λ and $x \neq 0$ such that $Ax = \lambda x$ with $\lambda + \lambda^* \geq 0$. But if X solves the Lyapunov equation

$$-x^* C^T C x = x^* A^T X x + x^* X A x = (\lambda + \lambda^*)\, x^* X x \geq 0$$

which implies $Cx = 0$, hence (A, C) not detectable.

Item 4. Assume that (A, C) is observable and A is Hurwitz. From Item 2. $X \succeq 0$. Assume X is not positive definite, that is, there exists $\bar{x} \neq 0$ such that $X\bar{x} = 0$. It follows that

$$0 = \bar{x}^* X \bar{x} = \int_0^\infty \bar{x}^* e^{A^T t} C^T C e^{At} \bar{x}\, dt = \int_0^\infty y^*(t) y(t)\, dt$$

which implies that response $y(t) = Ce^{At}\bar{x} = 0$ to a non null initial condition $x(0) = \bar{x}$ is null, which contradicts the hypothesis that (A, C) is observable.