1 Controllability and Observability

LTI system in state space

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t),\\
y(t) &= Cx(t).
\end{aligned}
$$

Observability Matrix:

$$
\mathcal{O}(A, C) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}.
$$

Controllability Matrix:

$$
\mathcal{C}(A, B) = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}.
$$

1.1 Determining initial conditions for analog simulation

Problem: Given *any* $y(0), \dot{y}(0), \cdots, y^{(n-1)}(0)$ compute $x(0), \cdots, x_n(0)$.

Use

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t), \\
\ddot{x}(t) &= A\dot{x}(t) + B\dot{u}(t) = A^2x(t) + ABu(t) + B\dot{u}(t), \\
\dddot{x}(t) &= A\ddot{x}(t) + B\ddot{u}(t) = A^3x(t) + A^2Bu(t) + AB\dot{u}(t) + B\ddot{u}(t),
\end{aligned}$$

and

$$\begin{aligned}
y(t) &= Cx(t), \\
\dot{y}(t) &= C\dot{x}(t) = CAx(t) + CBu(t), \\
\ddot{y}(t) &= C\ddot{x}(t) = CA^2x(t) + CABu(t) + CB\dot{u}(t),
\end{aligned}$$

to write

$$\mathcal{Y}(t) = \mathcal{O}(A, C)x(t) + \mathcal{T}\mathcal{U}(t)$$

where

$$\mathcal{Y}(t) = \begin{pmatrix} y(t) \\ \dot{y}(t) \\ \vdots \\ y^{(n-1)}(t) \end{pmatrix}, \qquad \mathcal{U}(t) = \begin{pmatrix} u(t) \\ \dot{u}(t) \\ \vdots \\ u^{(n-1)}(t) \end{pmatrix},$$

and

$$\mathcal{T} = \begin{bmatrix} 0 & & & 0 \\ CB & 0 & & \\ \vdots & & \ddots & \\ CA^{n-2}B & \cdots & CB & 0 \end{bmatrix} = \begin{bmatrix} H_0 & & & 0 \\ H_1 & H_0 & & \\ \vdots & & \ddots & \\ H_{n-1} & \cdots & H_1 & H_0 \end{bmatrix},$$

where $H_0 = 0$, and H_i, $i = 1, \ldots, n-1$, are the Markov parameters.

Using the fact $\mathcal{U}(0^-) = 0$ the initial conditions can be computed by solving

$$\mathcal{Y}(0^-) = \mathcal{O}(A, C)x(0^-).$$

1.1.1 Solving $\mathcal{Y}(0^-) = \mathcal{O}(A,C)x(0^-)$

From Linear Algebra

$\star$ When $\mathcal{Y}(0^-)$ is a linear combination of $\mathcal{O}(A,C)$ a solution $x(0^-)$ always exist!

$\star$ When $\mathcal{O}(A,C)$ does not have full-column rank there exists $\mathcal{Y}(0^-)$ for which $\mathcal{Y}(0^-) \neq \mathcal{O}(A,C)x(0^-) \quad \Rightarrow \quad$ NOT OBSERVABLE!

$\star$ When $\mathcal{O}(A,C)$ has full-column rank $\mathcal{O}(A,C)^T\mathcal{O}(A,C)$ is not singular.

$$\mathcal{O}(A,C)^T\mathcal{Y}(0^-) = \mathcal{O}(A,C)^T\mathcal{O}(A,C)x(0^-)$$
$$\Rightarrow \quad x(0^-) = \left[\mathcal{O}(A,C)^T\mathcal{O}(A,C)\right]^{-1}\mathcal{O}(A,C)^T\mathcal{Y}(0^-)$$

Side-effect: $x(0^-)$ is unique!
Proof by contradiction: Assume there exists $x_1(0^-) \neq x_2(0^-)$ such that $\mathcal{O}(A,C)x_1(0^-) = \mathcal{O}(A,C)x_2(0^-) = \mathcal{Y}(0^-)$. Then

$$\mathcal{O}(A,C)\left[x_1(0^-) - x_2(0^-)\right] = 0$$

which implies $\mathcal{O}(A,C)$ does not have full-column rank!

Theorem: The pair (A,C) is observable if and only if the observability matrix $\mathcal{O}(A,C)$ has full-column rank.

Proof: One missing point. Let

$$\mathcal{O}(A,C,i) := \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{i-1} \end{bmatrix}.$$

Rank of $\mathcal{O}(A,C,m) = \mathcal{O}(A,C,n)$ for all $m \geq n$.

1.2 Setting up initial conditions for analog simulation

Problem: Given $x(0) = 0$ and *any* $\bar{x}$, compute $u(t)$ such that $x(\bar{t}) = \bar{x}$ for some $\bar{t} > 0$.

Recall that

$$x^{(j)}(t) = A^j x(t) + \sum_{i=1}^{j} A^{j-i} B u^{(i-1)}(t)$$

Successively integrating both sides j times

$$\begin{aligned} x(t) &= \int_0^t \cdots \int_0^{\tau_2} x^{(j)}(\tau_1)\, d\tau_1 \cdots d\tau_j, \\ &= \int_0^t \cdots \int_0^{\tau_2} \sum_{i=1}^{j} A^{j-i} B u^{(i-1)}(\tau_1)\, d\tau_1 \cdots d\tau_j, \\ &= \sum_{i=1}^{j} A^{j-i} B \int_0^t \cdots \int_0^{\tau_2} u^{(i-1)}(\tau_1)\, d\tau_1 \cdots d\tau_j. \end{aligned}$$

so that

$$x(t) = \mathcal{C}(A, B) \int \mathcal{U}(t),$$

where

$$\int \mathcal{U}(t) = \int_0^t \cdots \int_0^{\tau_2} \mathcal{U}(t)\, d\tau_1 \cdots d\tau_n.$$

Given $x(\bar{t}) = \bar{x}$...

1.2.1 Solving $\bar{x} = \mathcal{C}(A, B) \int \mathcal{U}(\bar{t})$

From Linear Algebra

$\star$ When $\bar{x}$ is a linear combination of $\mathcal{C}(A, B)$ a solution $\int \mathcal{U}(\bar{t})$ always exist!

$\star$ When $\mathcal{C}(A, B)$ does not have full-row rank there exists $\bar{x}$ for which
$\bar{x} \neq \mathcal{C}(A, B) \int \mathcal{U}(\bar{t}) \quad \Rightarrow \quad$ NOT CONTROLLABLE!

$\star$ When $\mathcal{C}(A, B)$ has full-row rank $\mathcal{C}(A, B)\mathcal{C}(A, B)^T$ is not singular.

Searching for

$$\int \mathcal{U}(\bar{t}) = \mathcal{C}(A, B)^T \mathcal{Z},$$

we can find

$$\begin{aligned} \bar{x} = \mathcal{C}(A, B) \int \mathcal{U}(\bar{t}) &= \mathcal{C}(A, B)\mathcal{C}(A, B)^T \mathcal{Z}, \\ \Rightarrow \quad \mathcal{Z} &= \left[\mathcal{C}(A, B)\mathcal{C}(A, B)^T\right]^{-1} \bar{x}, \\ \Rightarrow \quad \int \mathcal{U}(\bar{t}) &= \mathcal{C}(A, B)^T \left[\mathcal{C}(A, B)\mathcal{C}(A, B)^T\right]^{-1} \bar{x} \end{aligned}$$

WARNING: In general, $\int \mathcal{U}(\bar{t})$ might not be unique!

Theorem: The pair (A, B) is controllable if and only if the controllability matrix $\mathcal{C}(A, B)$ has full-row rank.

Proof: Two missing points:

1) Rank of $\mathcal{C}(A, B, m) = \mathcal{C}(A, B, n)$ for all $m \geq n$; and

2) Can we find $u(t)$ such that $\int \mathcal{U}(\bar{t}) = \mathcal{C}(A, B)^T \left[\mathcal{C}(A, B)\mathcal{C}(A, B)^T\right]^{-1} \bar{x}$?

1.2.2 Solving for $u(t)$

Let $u(t) = \sum_{i=0}^{n-1} \alpha_i t^i$ so that

$$\mathcal{U}(t) = \begin{pmatrix} u(t) \\ \dot{u}(t) \\ \vdots \\ u^{(n-1)}(t) \end{pmatrix} = \begin{pmatrix} \sum_{i=0}^{n-1} \alpha_i t^i \\ \vdots \\ \sum_{i=j}^{n-1} \frac{i!}{(i-j)!} \alpha_i t^{i-j} \\ \vdots \\ (n-i)! \alpha_{n-i} \end{pmatrix},$$

$$= \underbrace{\begin{bmatrix} 1 & \cdots & t^j & \cdots & t^{n-1} \\ & \ddots & & & \vdots \\ & & j! & & \frac{(n-1)!}{(n-j-1)!} t^{n-j-1} \\ & & & \ddots & \vdots \\ 0 & & & & (n-1)! \end{bmatrix}}_{\mathbf{T}(t)} \underbrace{\begin{pmatrix} \alpha_0 \\ \vdots \\ \alpha_j \\ \vdots \\ \alpha_{n-i} \end{pmatrix}}_{\alpha},$$

Fact: $\mathbf{T}(t)$ is non singular for all t (why?) and

$$\int \mathbf{T}(t) = \int_0^t \cdots \int_0^{\tau_2} \begin{bmatrix} 1 & \cdots & \tau_1^j & \cdots & \tau_1^{n-1} \\ & \ddots & & & \vdots \\ & & j! & & \frac{(n-1)!}{(n-j-1)!} \tau_1^{n-j-1} \\ & & & \ddots & \vdots \\ 0 & & & & (n-1)! \end{bmatrix} d\tau_1 \cdots d\tau_n$$

$$= \begin{bmatrix} \frac{1}{n!} t^n & \cdots & \star & \cdots & \star \\ & \ddots & & & \vdots \\ & & \frac{j!}{n!} t^n & & \star \\ & & & \ddots & \vdots \\ 0 & & & & \frac{1}{n} t^n \end{bmatrix}.$$

is non singular for all $t > 0$ (why?). Therefore, for any $\bar{t} > 0$

$$\int \mathcal{U}(\bar{t}) = \left(\int \mathbf{T}(\bar{t}) \right) \alpha$$

and

$$\alpha = \left(\int \mathbf{T}(\bar{t}) \right)^{-1} \int \mathcal{U}(\bar{t}) = \left(\int \mathbf{T}(\bar{t}) \right)^{-1} \mathcal{C}(A,B)^T \left[\mathcal{C}(A,B) \mathcal{C}(A,B)^T \right]^{-1} \bar{x}.$$

1.3 The Cayley-Hamilton Theorem

Theorem: Let $d_A(s)$ be the characteristic polynomial of A. Then $d_A(A) = 0$.

Lemma: Let S and R be upper triangular matrices with the structure

$$S = \left[\begin{array}{c|c|c} 0 & S_2 & S_3 \\ \hline 0 & S_4 & S_5 \\ \hline 0 & 0 & S_6 \end{array}\right], \qquad R = \left[\begin{array}{c|c|c} R_1 & R_2 & R_3 \\ \hline 0 & 0 & R_5 \\ \hline 0 & 0 & R_6 \end{array}\right],$$

the product $T = SR$ has the structure

$$T = \left[\begin{array}{c|c|c} 0 & 0 & T_3 \\ \hline 0 & 0 & T_5 \\ \hline 0 & 0 & T_6 \end{array}\right],$$

Proof of the Lemma: Compute the product!

Proof of the Theorem: Write A as

$$A = TJT^{-1}$$

when J is upper-triangular with the eigenvalue λ_i in the ith position (e.g. in Jordan form). Then factor $d_A(s)$ as

$$d_A(s) = (s - \lambda_1)(s - \lambda_2) \cdots (s - \lambda_n)$$

so that

$$\begin{aligned} d_A(A) &= (A - \lambda_1 I)(A - \lambda_2 I) \cdots (A - \lambda_n I), \\ &= \left(TJT^{-1} - \lambda_1 I\right)\left(TJT^{-1} - \lambda_2 I\right) \cdots \left(TJT^{-1} - \lambda_n I\right), \\ &= T\left(J - \lambda_1 I\right)T^{-1}T\left(J - \lambda_2 I\right)T^{-1} \cdots T\left(J - \lambda_n I\right)T^{-1}, \\ &= T\left(J - \lambda_1 I\right)\left(J - \lambda_2 I\right) \cdots \left(J - \lambda_n I\right)T^{-1}, \\ &= T d_A(J) T^{-1}. \end{aligned}$$

Apply the above Lemma for $S = J - \lambda_1 I$ and $R = J - \lambda_2$. Apply it again for $S = (J - \lambda_1 I)(J - \lambda_2 I)$ and $R = (J - \lambda_3 I)$, and so on for the remaining terms to conclude that $d_A(J) = 0$. Therefore $d_A(A) = 0$.

1.4 Implications of the Cayley-Hamilton Theorem

$\star$ Any power of A^m for $m \geq n$ can be written as a linear combination of A_i, $i = 0, \ldots, n-1$. For example,

$$\begin{aligned} d_A(A) = A^n + a_1 A^{n-1} + a_2 A^{n-2} + \cdots + a_{n-1} A + a_n I = 0 \\ \Rightarrow \quad A^n = -a_1 A^{n-1} - a_2 A^{n-2} \cdots - a_{n-1} A - a_n I. \end{aligned}$$

$\star$ If $a_n \neq 0$ then A^{-1} can be written as a linear combination of A_i, $i = 0, \ldots, n-1$.

$$\begin{aligned} a_n I &= -\left(A^n + a_1 A^{n-1} + a_2 A^{n-2} + \cdots + a_{n-1} A\right), \\ &= A\left(A^{n-1} + a_1 A^{n-2} + a_2 A^{n-3} + \cdots + a_{n-1} I\right). \end{aligned}$$

Therefore

$$A^{-1} = -\left(A^{n-1} + a_1 A^{n-2} + a_2 A^{n-3} + \cdots + a_{n-1}\right)/a_n.$$

$\star$ Rank of $\mathcal{C}(A, B, m) = \mathcal{C}(A, B, n)$ for all $m \geq n$. For $m > n$

$$\mathcal{C}(A, B, m) = \begin{bmatrix} \mathcal{C}(A, B, n) & A^n B & \cdots & A^m B \end{bmatrix}.$$

Now use the Cayley-Hamilton theorem to write

$$A^m = p_m(I, A, \ldots, A^{n-1}), \qquad \forall m \geq n,$$

so that all columns of

$$A^m B = p_m(I, A, \ldots, A^{n-1}) B, \qquad \forall m \geq n,$$

are linear combinations of the columns of $\mathcal{C}(A, B, n)$.