1 Linear Time-Varying Systems

LTV system in state space

$$\dot{x}(t) = A(t)x(t) + B(t)u(t),$$
$$y(t) = C(t)x(t) + D(t)u(t).$$

1.1 Existence and uniqueness of solution

Differential equation:

$$\dot{x}(t) = f(x(t), t), \quad a \le t \le b.$$

Sufficient condition for existence and uniqueness of solution: $f(x, t)$ is *Lipschitz*, i.e.,

$$\|f(y(t), t) - f(x(t), t)\| \le k(t)\|y(t) - x(t)\|, \quad a \le t \le b,$$

where $k()$ is (piecewise) continuous.

For LTV $f(x, t) = A(t)x(t) + B(t)u(t)$ and

$$\begin{aligned} \|f(y(t), t) - f(x(t), t)\| &= \|A(t)y(t) + B(t)u(t) - A(t)x(t) - B(t)u(t)\|, \\ &= \|A(t)y(t) - A(t)x(t)\|, \\ &\le \|A(t)\|\|y(t) - x(t)\| \qquad \Rightarrow \qquad \text{Lipschitz!} \end{aligned}$$

Conclusion: LTV system has a solution and it is unique!

1.2 Solution to LTV

Scalar homogeneous equation

$$\dot{x}(t) = a(t)x(t), \qquad t \geq 0, \quad x(0) = x_0.$$

Separation of variables

$$\frac{1}{x}dx = a(t)dt.$$

Integrate on both sides

$$\int_{x(0)}^{x(t)} \frac{1}{x}dx = \int_0^t a(\tau)d\tau, \qquad \Rightarrow \quad x(t) = e^{\int_0^t a(\tau)d\tau}x(0).$$

In the matrix case, assume

$$x(t) = e^{\int_0^t A(\tau)d\tau}x(0),$$

and compute

$$\begin{aligned} \frac{d}{dt}x(t) &= \frac{d}{dt}e^{F(t)}x(0), \quad F(t) = \int_0^t A(\tau)d\tau, \\ &= \sum_{i=0}^{\infty} \frac{1}{i!}\frac{d}{dt}F^i(t)x(0). \end{aligned}$$

But

$$\begin{aligned} \frac{d}{dt}F^2(t) &= F(t)\frac{d}{dt}F(t) + [\frac{d}{dt}F(t)]F(t), \\ &= \int_0^t A(\tau)d\tau A(t) + A(t)\int_0^t A(\tau)d\tau \neq 2A(t)\int_0^t A(\tau)d\tau, \end{aligned}$$

since $A(t)$ and $\int_0^t A(\tau)d\tau$ do not necessarily commute!
Therefore

$$\frac{d}{dt}x(t) = \frac{d}{dt}e^{\int_0^t A(\tau)d\tau}x(0) \neq A(t)e^{\int_0^t A(\tau)d\tau}x(0) = A(t)x(t)$$

Remember that for LTI $A(t) = A$ and $\int_0^t A(\tau)d\tau = At$ commutes with A!

1.2.1 Equivalent transformations for LTV systems

LTV in state space

$$\begin{aligned}\dot{x}(t) &= A(t)x(t) + B(t)u(t),\\ y(t) &= C(t)x(t) + D(t)u(t).\end{aligned}$$

Let $P(t)$ be nonsingular for all t and define

$$x(t) = P(t)z(t)$$

such that

$$\begin{aligned}\dot{x}(t) = \dot{P}(t)z(t) + P(t)\dot{z}(t) &= A(t)P(t)z(t) + B(t)u(t),\\ \Rightarrow \quad P(t)\dot{z}(t) &= \left[A(t)P(t) - \dot{P}(t)\right] z(t) + B(t)u(t).\end{aligned}$$

Equivalent LTV system

$$\begin{aligned}\dot{z}(t) &= P(t)^{-1}\left[A(t)P(t) - \dot{P}(t)\right] x(t) + P(t)^{-1}B(t)u(t),\\ y(t) &= C(t)P(t)z(t) + D(t)u(t).\end{aligned}$$

1.2.2 Fundamental Matrix

$P(t)$ is called a *fundamental matrix* when

$$\dot{P}(t) = A(t)P(t), \qquad |P(t_0)| \neq 0.$$

If $P(t)$ is a fundamental matrix then

$$\dot{z}(t) = P(t)^{-1}B(t)u(t) \quad \Rightarrow \quad z(t) = z(t_0) + \int_{t_0}^{t} P(\tau)^{-1}B(\tau)u(\tau)d\tau.$$

and

$$\begin{aligned} x(t) &= P(t)z(t_0) + \int_{t_0}^{t} P(t)P(\tau)^{-1}B(\tau)u(\tau)d\tau, \\ &= P(t)P(t_0)^{-1}x(t_0) + \int_{t_0}^{t} P(t)P(\tau)^{-1}B(\tau)u(\tau)d\tau, \\ &= \Phi(t,t_0)x(t_0) + \int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)d\tau, \qquad \Phi(t,\tau) = P(t)P(\tau)^{-1}. \end{aligned}$$

1.2.3 State Transition Matrix

$\Phi(t,\tau)$ is called the *state transition matrix*

Properties

1) $\Phi(t,t) = I$,

2) $\Phi^{-1}(t,\tau) = \Phi(\tau,t)$,

3) $\Phi(t_1,t_2) = \Phi(t_1,t_0)\Phi(t_0,t_2)$.

4) $\frac{d}{dt}\Phi(t,\tau) = A\Phi(t,\tau), \qquad \Phi(\tau,\tau) = I$.

Proof:

1) $\Phi(t,t) = P(t)P^{-1}(t) = I$,

2) $\Phi^{-1}(t,\tau) = \left[P(t)P^{-1}(\tau)\right]^{-1} = P(\tau)P^{-1}(t) = \Phi(\tau,t)$,

3) $\Phi(t_1,t_2) = P(t_1)P^{-1}(t_2)P(t_2)P^{-1}(t_3) = P(t_1)P^{-1}(t_3) = \Phi(t_1,t_3)$.

4) $\frac{d}{dt}\Phi(t,\tau) = \frac{d}{dt}P(t)P(\tau)^{-1} = \dot{P}(t)P(\tau) = A(t)P(t)P(\tau) = A(t)\Phi(t,\tau)$.

1.2.4 Complete solution

$$
\begin{aligned}
y(t) &= C(t)x(t) + D(t)u(t), \\
&= C(t)\Phi(t,t_0)x(t_0) + \int_{t_0}^{t} C(t)\Phi(t,\tau)B(\tau)u(\tau)d\tau + D(t)u(t).
\end{aligned}
$$

For SIMO we have

$$
\begin{aligned}
y(t) &= C(t)\Phi(t,t_0)x(t_0) + \int_{t_0}^{t} \left[C(t)\Phi(t,\tau)B(\tau) + D(\tau)\delta(t-\tau)\right] u(\tau)d\tau, \\
&= C(t)\Phi(t,t_0)x(t_0) + \int_{t_0}^{t} h(t,\tau)u(\tau)d\tau,
\end{aligned}
$$

where

$$
h(t,\tau) = C(t)\Phi(t,\tau)B(\tau) + D(\tau)\delta(t-\tau)
$$

is the *impulse response*.

1.2.5 Floquet theory

There exists $\bar{P}(t)$ that transforms the LTV homogeneous system

$$\dot{x}(t) = A(t)x(t)$$

into the equivalent LTI homogeneous system

$$\dot{z}(t) = \bar{A}z(t)$$

We have already seen one case: $\bar{A} = 0$!

In general, for any $\bar{P}(t)$ nonsingular we have

$$\dot{z}(t) = \bar{P}(t)^{-1}\left[A(t)\bar{P}(t) - \dot{\bar{P}}(t)\right] z(t)$$

Defining

$$\bar{P}(t) = P(t)e^{-\bar{A}t},$$

where $P(t)$ is any fundamental matrix then

$$\begin{aligned}
\bar{P}(t)^{-1}\left[A(t)\bar{P}(t) - \dot{\bar{P}}(t)\right] &= e^{\bar{A}t}P^{-1}(t)\left[A(t)P(t)e^{-\bar{A}t} + P(t)e^{-\bar{A}t}\bar{A} - \dot{P}(t)e^{-\bar{A}t}\right], \\
&= e^{\bar{A}t}P^{-1}(t)P(t)e^{-\bar{A}t}\bar{A} \\
&\qquad + e^{\bar{A}t}P^{-1}(t)\left[A(t)P(t) - \dot{P}(t)\right]e^{-\bar{A}t}, \\
&= \bar{A}.
\end{aligned}$$

When $\bar{A} = 0$ then $\bar{P}(t) = P(t)$!

1.3 Example

LTV system

$$\dot{x} = \begin{bmatrix} 0 & 0 \\ t & 0 \end{bmatrix} x$$

is equivalent to equations

$$\begin{cases} \dot{x}_1 = 0, \\ \dot{x}_2 = t x_1. \end{cases} \quad \Rightarrow \quad \begin{cases} x_1(t) = x_1(t_0) \\ x_2(t) = x_2(t_0) + \dfrac{1}{2}(t^2 - t_0^2) x_1(t_0) \end{cases}$$

Fundamental matrix for $t_0 = 0$

$$\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \Rightarrow \qquad \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{2}t^2 \end{pmatrix},$$
$$\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \Rightarrow \qquad \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

so that

$$P(t) = \begin{bmatrix} 1 & 0 \\ \frac{1}{2}t^2 & 1 \end{bmatrix}.$$

State transition matrix

$$\begin{aligned} \Phi(t, t_0) &= P(t) P^{-1}(t_0), \\ &= \begin{bmatrix} 1 & 0 \\ \frac{1}{2}t^2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{1}{2}t_0^2 & 1 \end{bmatrix}^{-1}, \\ &= \begin{bmatrix} 1 & 0 \\ \frac{1}{2}t^2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{2}t_0^2 & 1 \end{bmatrix}, \\ &= \begin{bmatrix} 1 & 0 \\ -\frac{1}{2}(t^2 - t_0^2) & 1 \end{bmatrix}. \end{aligned}$$