1 Zeros of LTI Systems

1.1 SISO Systems

$$H(s) = \frac{N(s)}{D(s)} = C(sI - A)^{-1}B + D$$

Definition: $s_0 \in \mathbb{C}$ is a zero if $N(s_0) = 0$.

Assumption: Matrix

$$Q(s) = \begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix}$$

is nonsingular for some $s \in \mathbb{C}$.

Theorem: Assume s_0 is not a pole. Then s_0 is a zero of $H(s) = C(sI - A)^{-1}B + D$ if and only if $Q(s_0)$ is singular.

From Linear Algebra:
⋆ Matrix $V = XYZ$ is nonsingular iff X, Y and Z are nonsingular.
⋆ The triangular matrix $\begin{bmatrix} A & 0 \\ B & C \end{bmatrix}$ is nonsingular iff A and C are nonsingular.

Proof: Verify that

$$\begin{aligned} Q(s) &= \begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix} \\ &= \begin{bmatrix} sI - A & 0 \\ C & I \end{bmatrix} \begin{bmatrix} (sI - A)^{-1} & 0 \\ 0 & C(sI - A)^{-1}B + D \end{bmatrix} \begin{bmatrix} sI - A & -B \\ 0 & I \end{bmatrix} \\ &= \begin{bmatrix} sI - A & 0 \\ C & I \end{bmatrix} \begin{bmatrix} (sI - A)^{-1} & 0 \\ 0 & H(s) \end{bmatrix} \begin{bmatrix} sI - A & -B \\ 0 & I \end{bmatrix}. \end{aligned}$$

If s_0 is not a pole then $(s_0 I - A)$ is nonsingular. Therefore

$$(s_0 I - A)^{-1}, \qquad \begin{bmatrix} s_0 I - A & 0 \\ C & I \end{bmatrix}, \qquad \text{and} \qquad \begin{bmatrix} s_0 - A & -B \\ 0 & I \end{bmatrix}$$

are nonsingular. Therefore $Q(s_0)$ is singular iff $H(s_0) = 0$, i.e., iff s_0 is a zero.

WARNING: What if s_0 is a pole?

1.2 MIMO Systems

Extension to MIMO Systems: s_0 is such that $H(s)$ looses rank.
If H is square, this means $H(s_0)$ is singular.

Normal rank: $\text{normalrank}\, Q(s) := \max_{s \in \mathbb{C}} \{\text{rank}\, Q(s)\}$.

Definition: $s_0 \in \mathbb{C}$ is a zero if $\text{rank}\, Q(s_0) < \text{normalrank}\, Q(s)$.

Assumption: Matrix $Q(s)$ has full normal rank.

Theorem: Assume that $Q(s)$ has full-column normal rank. Then s_0 is a zero of $H(s) = C(sI - A)^{-1}B + D$ if and only if there exists $x \neq 0$, u such that

$$Q(s_0) \begin{pmatrix} x \\ u \end{pmatrix} = \begin{bmatrix} s_0 I - A & -B \\ C & D \end{bmatrix} \begin{pmatrix} x \\ u \end{pmatrix} = 0. \tag{1}$$

Proof:
Sufficiency: If $\exists\ s_0 \in \mathbb{C}$ and $x \neq 0$, u such that (1) holds, then $Q(s_0)$ does not have full-collumn rank, and s_0 is a zero according to the definition.
Necessity: If $\exists\ s_0 \in \mathbb{C}$ is a zero then there exists $\begin{pmatrix} x \\ u \end{pmatrix} \neq 0$ such that (1) holds. All we have to prove is that $x \neq 0$. Assume (1) holds with $x = 0$. Then

$$Q(s_0) \begin{pmatrix} 0 \\ u \end{pmatrix} = \begin{bmatrix} -B \\ D \end{bmatrix} u = 0.$$

But since $Q(s)$ has full-collumn normal rank this implies $u = 0$ and $\begin{pmatrix} x \\ u \end{pmatrix} \neq 0$, which is a contradiction.

Theorem: Assume that $Q(s)$ has full-row normal rank. Then s_0 is a zero of $H(s) = C(sI - A)^{-1}B + D$ if and only if there exists $y \neq 0$, v such that

$$\begin{pmatrix} y^* & v^* \end{pmatrix} Q(s_0) = \begin{pmatrix} y^* & v^* \end{pmatrix} \begin{bmatrix} s_0 I - A & -B \\ C & D \end{bmatrix} = 0.$$

1.2.1 Example 1

Consider

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1 \end{bmatrix}, \qquad D = 0,$$

and compute

$$H(s) = \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} s+1 & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{2s} \\ \frac{1}{s} & \frac{1}{s} \end{bmatrix}.$$

$H_{ij}(s)$ has no zeros for all i, j! However

$$\operatorname{normalrank} Q(s) = \operatorname{rank} Q(2) = 5,$$

and

$$\operatorname{rank} Q(1) = 4 < \operatorname{normalrank} Q(s).$$

Indeed

$$Q(1) \begin{pmatrix} x \\ u \end{pmatrix} = \left[\begin{array}{ccc|cc} 2 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ \hline 1 & 0 & 1/2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{array}\right] \left(\begin{array}{c} 1 \\ 2 \\ -2 \\ \hline 2 \\ -2 \end{array}\right) = 0.$$

Conclusion: $s_0 = 1$ is a zero of $H(s)$!

1.2.2 Example 2

Consider

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad C = \begin{bmatrix} -1 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & 1 \end{bmatrix}.$$

and compute

$$H(s) = \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} s+1 & 0 \\ 0 & s \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{s}{s+1} & \frac{s+1}{s} \end{bmatrix}$$

$s = 0$ is a zero of $H_{11}(s)$ and $s = -1$ is a zero of $H_{12}(s)$! However

$$\operatorname{normalrank} Q(s) = \operatorname{rank} Q(2) = 3.$$

but there exists no $s_0 \in \mathbb{C}$ such that $\operatorname{rank} Q(s_0) < 3$.
Conclusion: $H(s)$ has no zeros!

2 Summary for LTI Systems

... and brief introduction to Matlab.

2.1 Example

Transfer function:

$$H(s) = \frac{s+1}{s^2+5s+6}$$

Observer state space cannonical realization:

$$\dot{x} = \begin{bmatrix} -5 & 1 \\ -6 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u,$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

Poles:

$$\begin{vmatrix} s+5 & -1 \\ 6 & s \end{vmatrix} = 0, \quad s^2 + 5s + 6 = (s+2)(s+3) = 0$$

Distinct eigenvalues $\lambda_1 = -2$, $\lambda_2 = -3$, $\quad \Rightarrow \quad A$ is diagonalizable.
Eigenvectors:

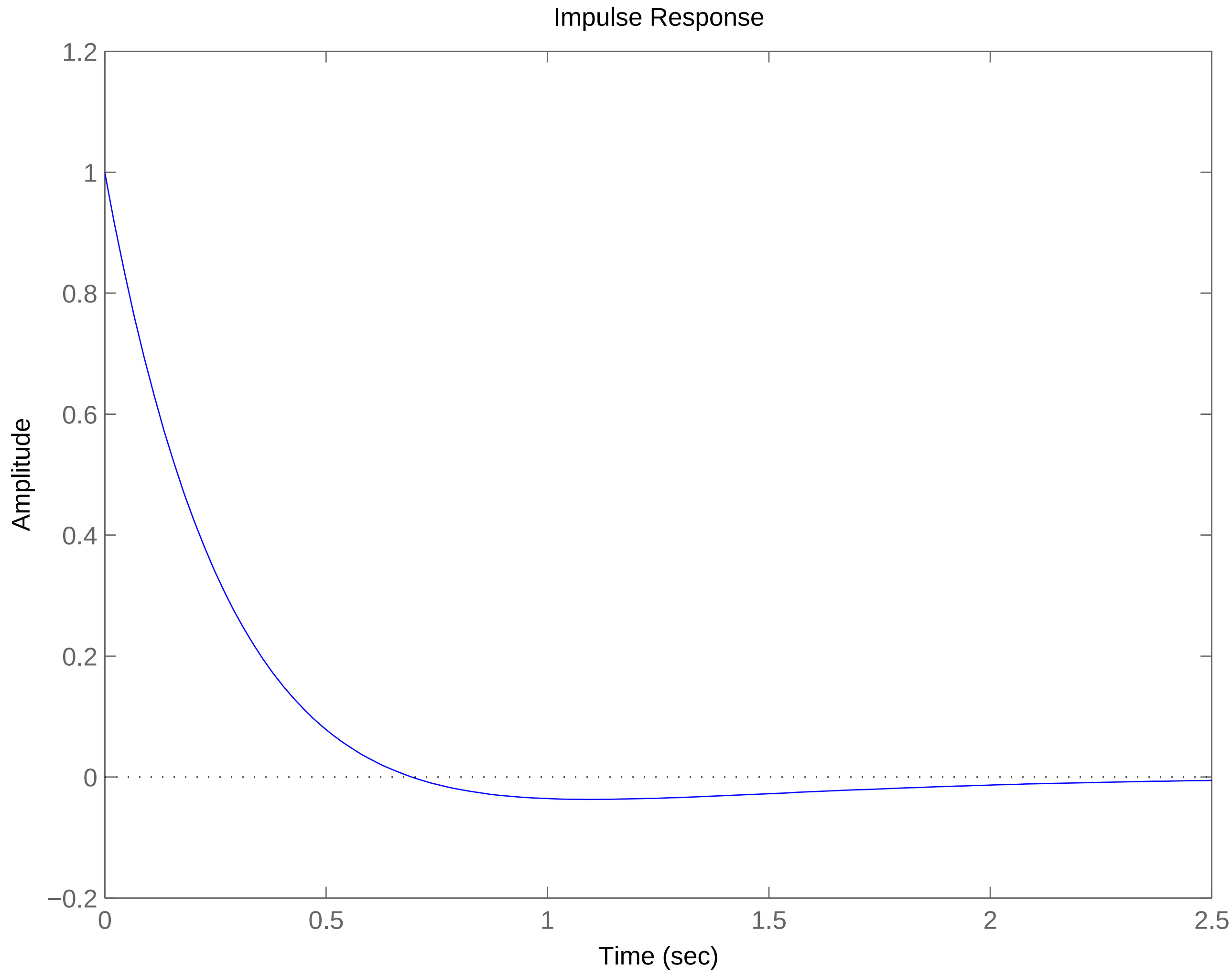
$$\left.\begin{aligned} (-2I - A)v_1 &= \begin{bmatrix} 3 & -1 \\ 6 & -2 \end{bmatrix} v_1 = 0, \quad v_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \\ (-3I - A)v_2 &= \begin{bmatrix} 2 & -1 \\ 6 & -3 \end{bmatrix} v_2 = 0, \quad v_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \end{aligned}\right\} \quad \Rightarrow \quad \begin{aligned} T &= \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}, \\ T^{-1} &= \begin{bmatrix} -2 & 1 \\ 3 & -1 \end{bmatrix}. \end{aligned}$$

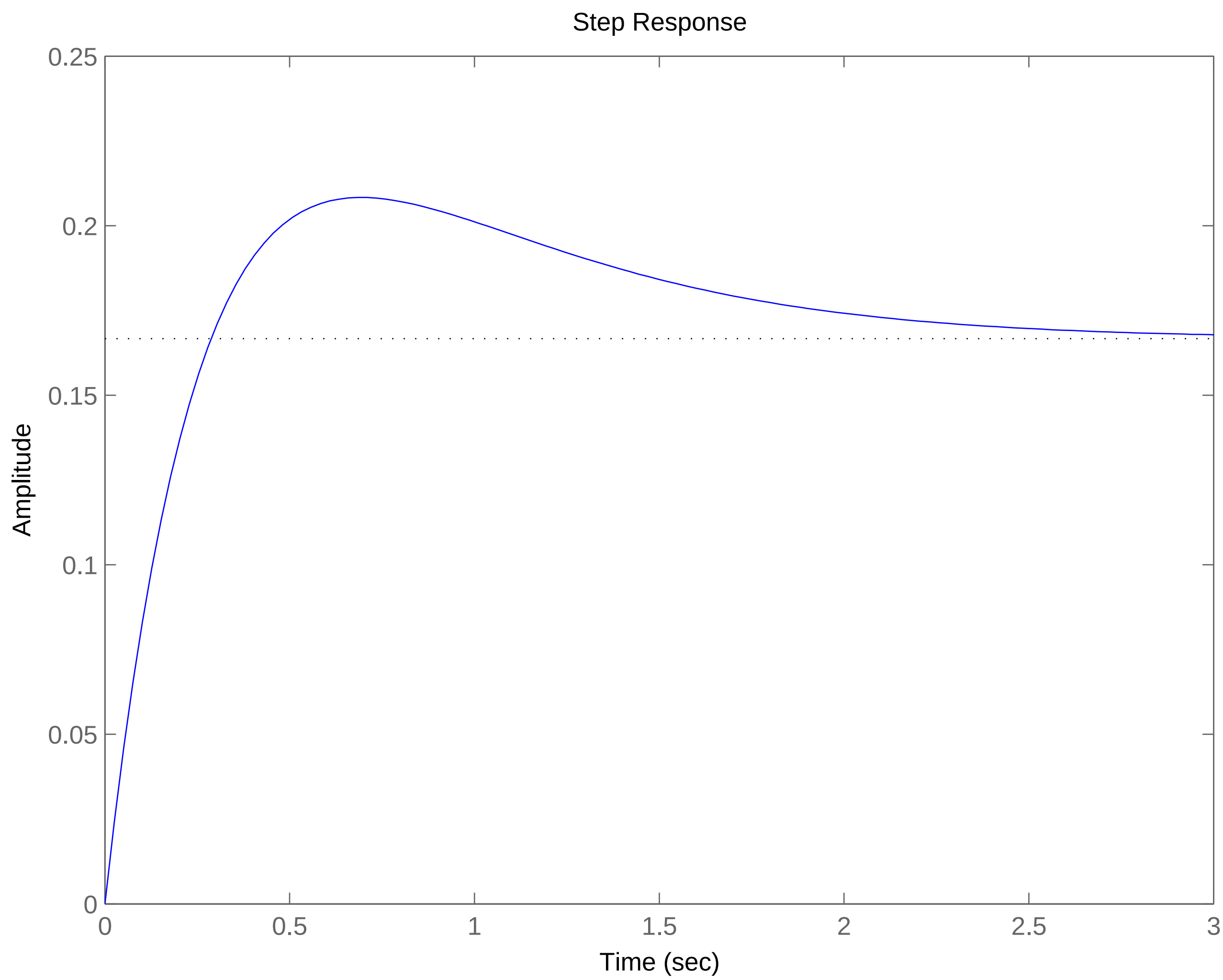
Diagonal state space realization:

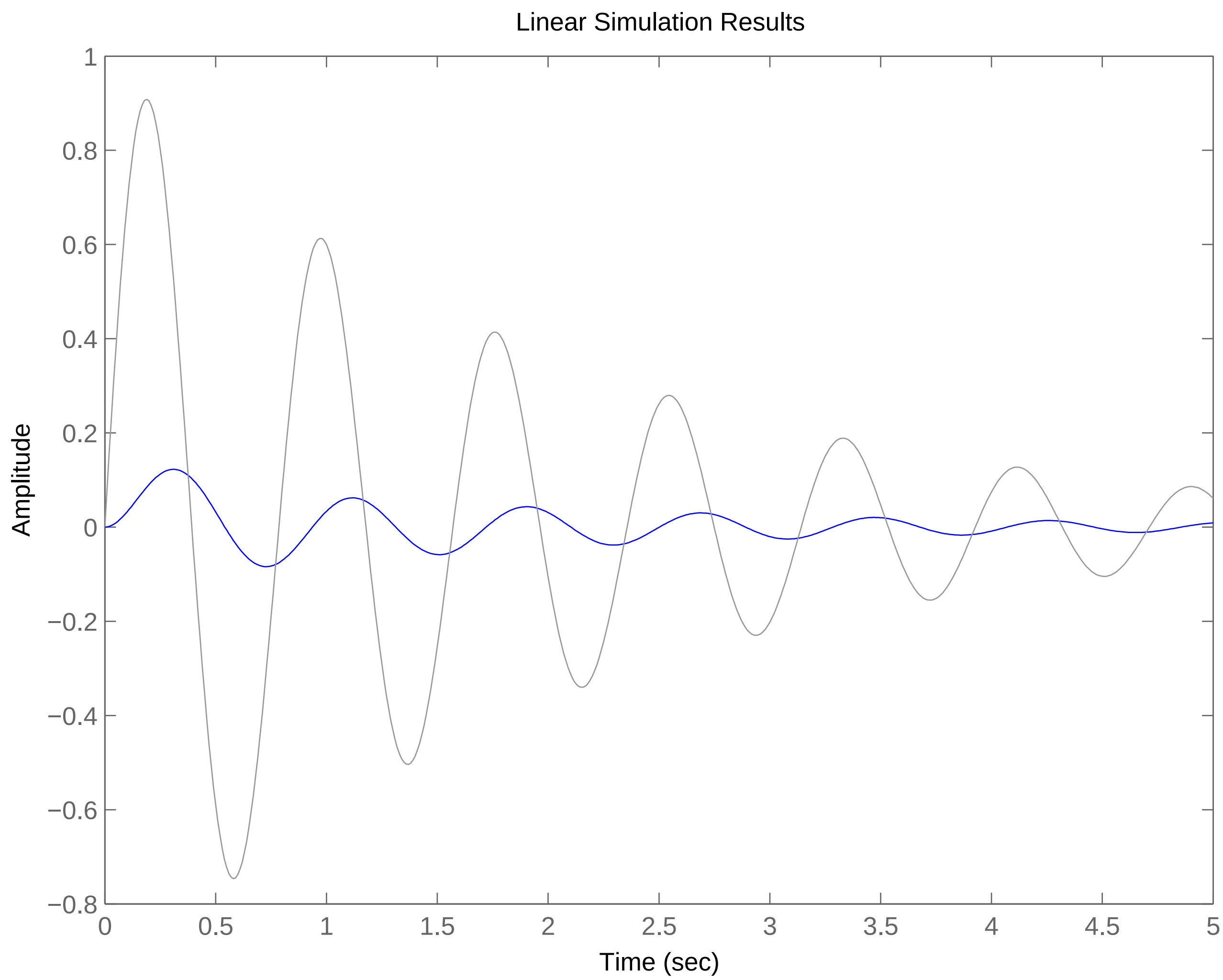
$$\dot{x} = T^{-1}ATx + T^{-1}Bu = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} x + \begin{bmatrix} -1 \\ 2 \end{bmatrix} u,$$
$$y = CTx = \begin{bmatrix} 1 & 1 \end{bmatrix} x.$$

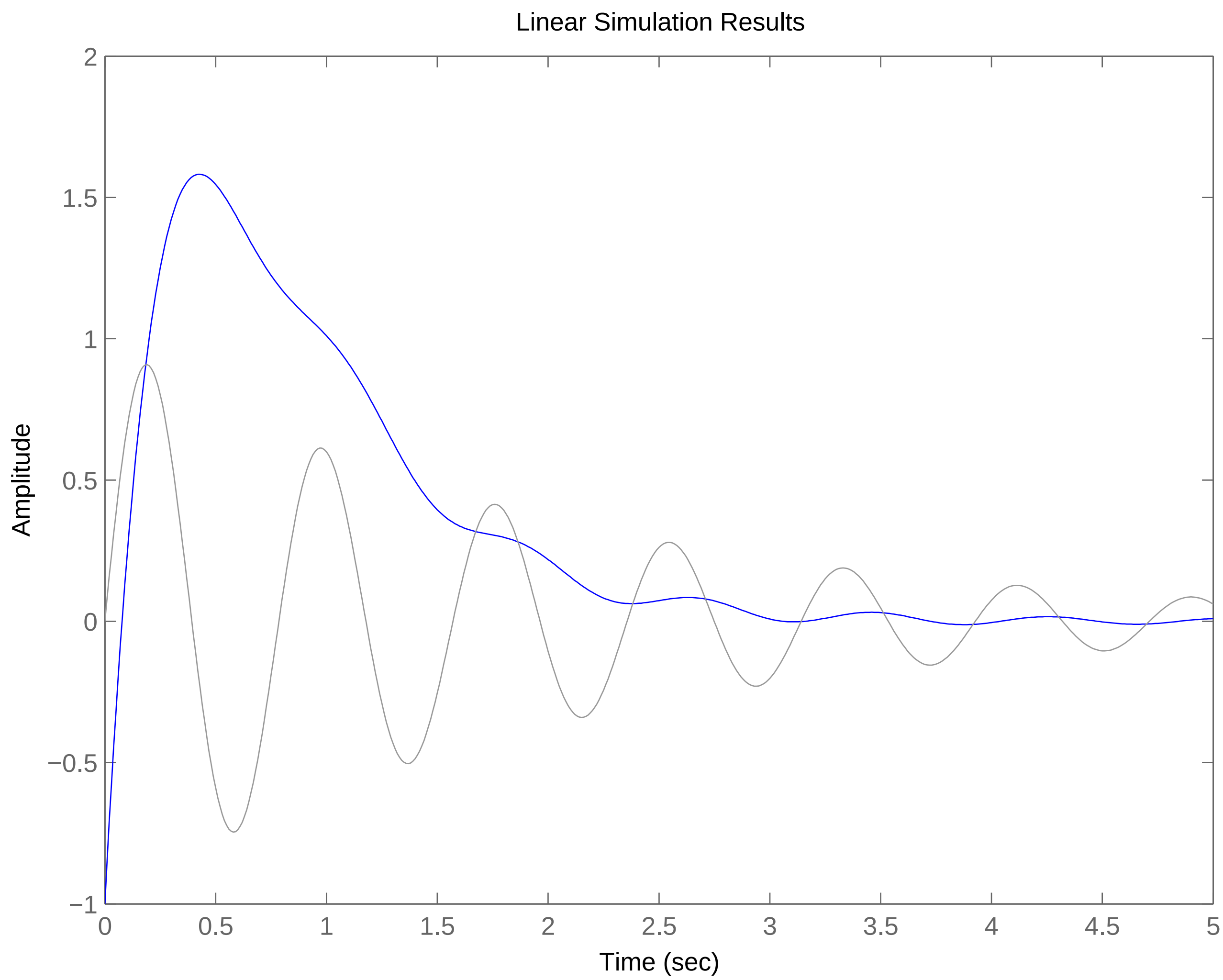
Time response:

$$\begin{aligned} y(t) &= Ce^{At}x_0 + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau, \\ &= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{-3t} \end{bmatrix} x_0 + \int_0^t \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} e^{-2(t-\tau)} & 0 \\ 0 & e^{-3(t-\tau)} \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} u(\tau)d\tau, \\ &= \begin{bmatrix} e^{-2t} & e^{-3t} \end{bmatrix} x_0 + \int_0^t \left(2e^{-3(t-\tau)} - e^{-2(t-\tau)}\right) u(\tau)d\tau. \end{aligned}$$


Impulse Response
Amplitude
Time (sec)
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
0
0.5
1
1.5
2
2.5


Step Response
Amplitude
Time (sec)
0
0.05
0.1
0.15
0.2
0.25
0
0.5
1
1.5
2
2.5
3


Linear Simulation Results
Amplitude
Time (sec)
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5


Linear Simulation Results
Amplitude
2
1.5
1
0.5
0
−0.5
−1
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
Time (sec)