1 Solution to Linear Time-Invariant Systems

1.1 Scalar equation

Homogeneous equation

$$\frac{dx}{dt} = ax, \quad x(0) = x_0$$

Separation of variables

$$\frac{1}{x}\,dx = a\,dt$$

Integrating both sides

$$\ln x\Big|_{x(0)}^{x(t)} = \int_{x(0)}^{x(t)} \frac{1}{x} dx = \int_0^t a d\tau = a\tau\Big|_0^t$$

Solution

$$\ln x(t) - \ln x(0) = \ln \frac{x(t)}{x(0)} = at \quad \Rightarrow \quad x(t) = e^{at} x(0) = e^{at} x_0$$

1.2 Scalar system

LTI scalar system

$$\begin{aligned}\dot{x}(t) &= ax(t) + bu(t), \\ y(t) &= cx(t) + du(t).\end{aligned} \tag{1}$$

Useful properties

$$\frac{d}{dt}e^{at} = ae^{at}, \qquad \frac{d}{dt}\left[e^{-at}x(t)\right] = e^{-at}\dot{x}(t) - ae^{-at}x(t).$$

Multiply (1) by e^{-at} on both sides

$$e^{-at}\dot{x}(t) - ae^{-at}x(t) = e^{-at}bu(t) \quad \Rightarrow \quad \frac{d}{dt}\left[e^{-at}x(t)\right] = e^{-at}bu(t)$$

Integrate on both sides

$$e^{-at}x(t)\Big|_0^t = \int_0^t \frac{d}{d\tau}\left[e^{-a\tau}x(\tau)\right] d\tau = \int_0^t e^{-a\tau}bu(\tau)d\tau$$

Solution

$$e^{-at}x(t) - x(0) = \int_0^t e^{-a\tau}bu(\tau)d\tau$$

$$\Rightarrow \quad x(t) = e^{at}x(0) + \int_0^t e^{a(t-\tau)}bu(\tau)d\tau$$

Substitute into the output equation

$$y(t) = ce^{at}x(0) + \int_0^t ce^{a(t-\tau)}bu(\tau)d\tau + du(t)$$

1.3 The matrix exponential

Taylor series (convergent for all x finite!)

$$e^x = \sum_{i=0}^{\infty} \frac{1}{i!} x^i = 1 + x + \frac{1}{2!} x^2 + \cdots + \frac{1}{n!} x^n + \cdots$$

Generalization for matrices

$$e^X = \sum_{i=0}^{\infty} \frac{1}{i!} X^i.$$

Properties:

1) $e^0 = I$,

2) $e^{A+B} = e^A e^B$, when $AB = BA$,

3) $[e^X]^{-1} = e^{-X}$.

Proof:

1) By definition.

2) Use the definition to compute

$$\begin{aligned} e^A e^B &= \left(I + A + \frac{1}{2!} A^2 + \frac{1}{3!} A^3 + \cdots \right) \left(I + B + \frac{1}{2!} B^2 + \frac{1}{3!} B^3 + \cdots \right) \\ &= I + (A+B) + \frac{1}{2!} \left(A^2 + 2AB + B^2 \right) \\ &\quad + \frac{1}{3!} \left(A^3 + 3A^2 B + 3AB^2 + B^3 \right) + \cdots \end{aligned}$$

and compare with

$$\begin{aligned} e^{A+B} &= I + (A+B) + \frac{1}{2!}(A+B)^2 + \frac{1}{3!}(A+B)^3 + \cdots \\ &= I + (A+B) + \frac{1}{2!} \left(A^2 + AB + BA + B^2 \right) \\ &\quad + \frac{1}{3!} \left(A^3 + BA^2 + ABA + B^2 A + A^2 B + BAB + AB^2 + B^3 \right) + \cdots \end{aligned}$$

3) X commutes with X. Then from 1) and 2)

$$I = e^{X-X} = e^X e^{-X} \quad \Rightarrow \quad [e^X]^{-1} = e^{-X}.$$

1.4 Time dependent matrix exponential

We are interested in

$$e^{At} = \sum_{i=0}^{\infty} \frac{1}{i!} A^i t^i.$$

Properties:

4) $e^{A(t_1+t_2)} = e^{At_1} e^{At_2}$,

5) $\frac{d}{dt} e^{At} = A e^{At} = e^{At} A$.

6) $\frac{d}{dt}\left[e^{-At} x(t)\right] = e^{-At}\, \dot{x}(t) - e^{-At} A x(t)$.

Proof:

4) At_1 and At_2 commute.

5) Use definition

$$\begin{aligned}
\frac{d}{dt} e^{At} &= \frac{d}{dt}\left(\sum_{i=0}^{\infty} \frac{1}{i!} A^i t^i\right), \\
&= \sum_{i=0}^{\infty} \frac{1}{(i-1)!} A^i t^{i-1}, \\
&= A\left(\sum_{i=0}^{\infty} \frac{1}{(i-1)!} t^{i-1} A^{i-1}\right) = A e^{At}, \\
&= \left(\sum_{i=0}^{\infty} \frac{1}{(i-1)!} t^{i-1} A^{i-1}\right) A = e^{At} A.
\end{aligned}$$

6) Chain rule

$$\begin{aligned}
\frac{d}{dt}\left[e^{-At} x(t)\right] &= e^{-At} \frac{d}{dt} x(t) + \frac{d}{dt}\left[e^{-At}\right] x(t), \\
&= e^{-At}\, \dot{x}(t) - e^{-At} A x(t).
\end{aligned}$$

1.5 Complete solution

LTI System

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t),\\ y(t) &= Cx(t) + Du(t),\end{aligned}$$

Multiply by e^{-At} on both sides

$$e^{-At}\,\dot{x}(t) - e^{-At}Ax(t) = e^{-At}Bu(t) \quad \Rightarrow \quad \frac{d}{dt}\left[e^{-At}x(t)\right] = e^{-At}Bu(t)$$

Integrate on both sides

$$e^{-At}x(t)\Big|_0^t = \int_0^t \frac{d}{d\tau}\left[e^{-A\tau}x(\tau)\right]d\tau = \int_0^t e^{-A\tau}Bu(\tau)d\tau$$

Solution

$$e^{-At}x(t) - x(0) = \int_0^t e^{-A\tau}Bu(\tau)d\tau$$

$$\Rightarrow \quad x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

Substitute on the output equation

$$y(t) = Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$

1.6 Impulse response (SIMO)

Recall that

$$\delta(t-\tau) = \delta(\tau - t)$$

Therefore

$$\begin{aligned}\int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t) &= \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + \int_0^t D\delta(\tau - t)u(\tau)d\tau, \\ &= \int_0^t \big[\, \underbrace{Ce^{A(t-\tau)}B + D\delta(t-\tau)}_{h(t-\tau)} \,\big]\, u(\tau)d\tau,\end{aligned}$$

where

$$h(t) = Ce^{At}B + D\delta(t)$$

is the *impulse response* of the system.

System response:

$$y(t) = \underbrace{Ce^{At}x(0)}_{\text{Initial conditions response}} + \underbrace{\int_0^t h(t-\tau)u(\tau)d\tau}_{\text{Input response}}.$$

Repeat the above for each input for MIMO systems.

1.7 Application: discretization

LTI continuous-time system

$$\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t) + Du(t),$$

If $u(t)$ comes from a computer

$$u(t) = u(k) := u(kh), \quad \forall\, kh \le t < (k+1)h$$

h : update period

The solution $x(t)$ of the LTI continuous-time system at $t = kh$ is

$$x(k) := x(kh) = e^{Akh}x(0) + \int_0^{kh} e^{A(kh-\tau)}Bu(\tau)d\tau$$

and at $t = (k+1)h$

$$\begin{aligned} x(k+1) := x[(k+1)h] &= e^{A(k+1)h}x(0) + \int_0^{(k+1)h} e^{A[(k+1)h-\tau]}Bu(\tau)d\tau, \\ &= e^{Ah}\left[e^{Akh}x(0) + \int_0^{kh} e^{A(kh-\tau)}Bu(\tau)d\tau\right] \\ &\qquad + \int_{kh}^{(k+1)h} e^{A[(k+1)h-\tau]}Bu(\tau)d\tau, \\ &= e^{Ah}x(k) + \int_{kh}^{(k+1)h} e^{A[(k+1)h-\tau]}Bu(\tau)d\tau. \end{aligned}$$

Since $u(t) = u(k), \quad \forall\, kh \le t < (k+1)h$

$$x(k+1) = \underbrace{e^{Ah}}_{F}\, x(k) + \underbrace{\int_{kh}^{(k+1)h} e^{A[(k+1)h-\tau]}\, d\tau B}_{G}\, u(k).$$

Note that

$$G = \int_{kh}^{(k+1)h} e^{A[(k+1)h-\tau]}\, d\tau B$$

and for $\sigma := (k+1)h - \tau$

$$G = -\int_{h}^{0} e^{A\sigma}\, d\sigma B = \int_{0}^{h} e^{A\sigma}\, d\sigma B$$

Discretized LTI system

$$\begin{aligned} x(k+1) &= Fx(k) + Gu(k), \\ y(k) &= Cx(k) + Du(k). \end{aligned}$$

where

$$\begin{aligned} F &= e^{Ah}, \\ G &= \int_{0}^{h} e^{A\sigma}\, d\sigma B. \end{aligned}$$

2 How to compute the matrix exponential?

2.1 Eigenvalues and Eigenvectors

Definition: $\lambda \in \mathbb{C}$ is an eigenvalue of $A \in \mathbb{R}^{n\times n}$ (or $\in \mathbb{C}^{n\times n}$) if there exists $v \in \mathbb{C}^n$ such that

$$Av = \lambda v, \quad v \neq 0.$$

If so, v is an eigenvector associated with λ.

From linear algebra

$$Av = \lambda v \quad \Leftrightarrow \quad (\lambda I - A)v = 0.$$

and $v \neq 0$ if and only if

$$d(\lambda) = \det(\lambda I - A) = 0.$$

Conclusion:
Eigenvalues of A are the roots of the characteristic polynomial of A.
$\Rightarrow \quad A$ has n eigenvalues.

2.2 Diagonalizable matrix

Definition: Matrix $A \in \mathbb{R}^{n\times n}$ (or $\in \mathbb{C}^{n\times n}$) is said to be *diagonalizable* if it is similar to a diagonal matrix, i.e., if there exists a nonsingular matrix T such that $T^{-1}AT$ is diagonal.

Theorem: Matrix $A \in \mathbb{R}^{n\times n}$ (or $\in \mathbb{C}^{n\times n}$) is diagonalizable if and only if it has n linearly independent eigenvectors.

From linear algebra
$\star$ The vectors $v_1, \ldots, v_n \in \mathbb{C}^n$ are *linearly dependent* if there exists $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ not all null such that

$$\sum_{i=1}^{n} \alpha_i v_i = 0.$$

Otherwise, they are *linearly independent*.

$\star$ If $v_1, \ldots, v_n \in \mathbb{C}^n$ are linearly independent then the matrix

$$T = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \tag{2}$$

is nonsingular. Proof: there exists no

$$x = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \neq 0$$

such that $Tx = 0$.

Proof of the Theorem:
Let $v_1, \ldots, v_n$ be n linearly independent eigenvectors of A and build T as in (2). Therefore

$$\begin{bmatrix} Av_1 & \cdots & Av_n \end{bmatrix} = AT = T\Lambda = \begin{bmatrix} \lambda_1 v_1 & \cdots & \lambda_n v_n \end{bmatrix},$$

where

$$\Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}.$$

As T is nonsingular

$$T^{-1}AT = \Lambda.$$

2.3 Matrix function

Let $f(x) : \mathbb{C} \to \mathbb{C}$ be a continuous scalar-valued function.

Definition: If X is a diagonalizable matrix, i.e., $X = T\Lambda T^{-1}$, then

$$f(X) = T \begin{bmatrix} f(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & f(\lambda_n) \end{bmatrix} T^{-1}.$$

If f has a power series expansion

$$f(x) = \sum_{i=0}^{\infty} \alpha_i x^i$$

then

$$\begin{aligned} f(X) &= T \begin{bmatrix} f(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & f(\lambda_n) \end{bmatrix} T^{-1}, \\ &= T \left(\sum_{i=0}^{\infty} \alpha_i \Lambda^i \right) T^{-1}, \\ &= \sum_{i=0}^{\infty} \alpha_i \left(T \Lambda^i T^{-1} \right), \\ &= \sum_{i=0}^{\infty} \alpha_i X^i. \end{aligned}$$

Compare the above with the formula previously used for e^X!