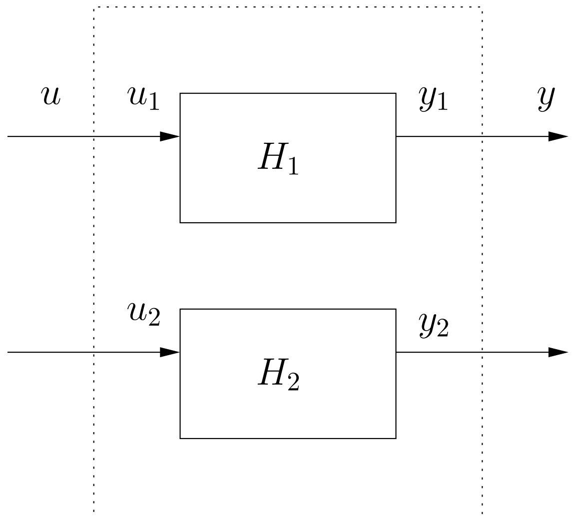
1 System Connections

$$H_1 : \begin{cases} \dot{x}_1 = A_1 x_1 + B_1 u_1, \\ y_1 = C_1 x_1 + D_1 u_1, \end{cases} \qquad H_2 : \begin{cases} \dot{x}_2 = A_2 x_2 + B_2 u_2, \\ y_2 = C_2 x_2 + D_2 u_2 \end{cases}$$

1.1 Parallel



Transfer function

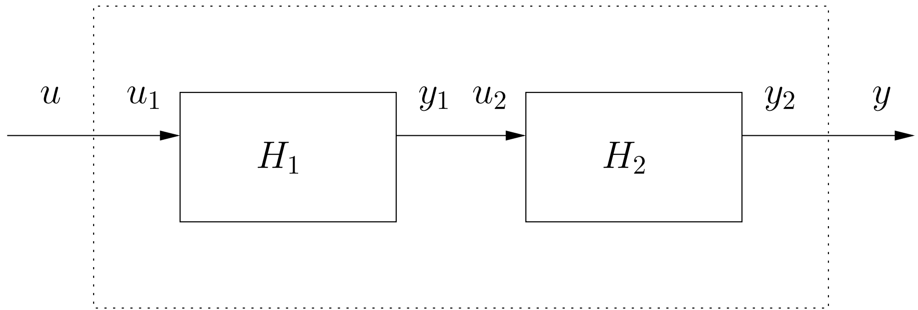
$$H(s) = \begin{bmatrix} H_1(s) & 0 \\ 0 & H_2(s) \end{bmatrix}$$

State space

$$\dot{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

1.2 Series



Transfer function

$$H(s) = H_2(s)H_1(s)$$

Use

$$u_2 = y_1 = C_1 x_1 + D_1 u_1, \qquad u = u_1, \qquad y = y_2,$$

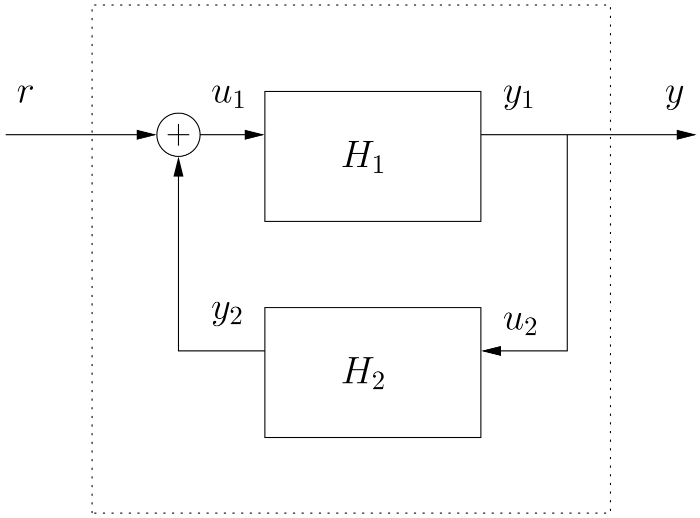
and rewrite H_2

$$\begin{aligned} \dot{x}_2 &= A_2 x_2 + B_2 C_1 x_1 + B_2 D_1 u, \\ y = y_2 &= C_2 x_2 + D_2 C_1 x_1 + D_2 D_1 u \end{aligned}$$

State space

$$\begin{aligned} \dot{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}} &= \begin{bmatrix} A_1 & 0 \\ B_2 C_1 & A_2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{bmatrix} B_1 \\ B_2 D_1 \end{bmatrix} u, \\ y &= \begin{bmatrix} D_2 C_1 & C_2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + D_2 D_1 u. \end{aligned}$$

1.3 Feedback (assumption: $D_1 = 0$)



Note that

$$
\begin{aligned}
U_1(s) &= R(s) + Y_2(s), \\
&= R(s) + H_2(s)U_2(s), \\
&= R(s) + H_2(s)Y_1(s), \\
&= R(s) + H_2(s)H_1(s)U_1(s).
\end{aligned}
$$

and

$$[I - H_2(s)H_1(s)]U_1(s) = R(s) \quad \Rightarrow \quad U_1(s) = [I - H_2(s)H_1(s)]^{-1}R(s).$$

Therefore

$$Y(s) = Y_1(s) = H_1(s)U_1(s) = H_1(s)[I - H_2(s)H_1(s)]^{-1}R(s).$$

Transfer function

$$H(s) = H_1(s)[I - H_2(s)H_1(s)]^{-1}.$$

Use

$$u_1 = y_2 + r, \qquad u_2 = y_1, \qquad y = y_1$$

Due to the assumption $D_1 = 0$

$$u_2 = y_1 = C_1 x_1.$$

Note that

$$\begin{aligned} u_1 &= y_2 + r, \\ &= C_2 x_2 + D_2 u_2 + r, \\ &= D_2 C_1 x_1 + C_2 x_2 + r \end{aligned}$$

Now rewrite H_1

$$\begin{aligned} \dot{x}_1 &= (A_1 + B_1 D_2 C_1) x_1 + B_1 C_2 x_2 + B_1 r, \\ y = y_1 &= C_1 x_1. \end{aligned}$$

and H_2

$$\dot{x}_2 = B_2 C_1 x_1 + A_2 x_2,$$

State space

$$\begin{aligned} \dot{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}} &= \begin{bmatrix} A_1 + B_1 D_2 C_1 & B_1 C_2 \\ B_2 C_1 & A_2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} r, \\ y &= \begin{bmatrix} C_1 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \end{aligned}$$

2 Properties of LTI systems

2.1 Transfer function

$$
\begin{aligned}
H(s) &= C(sI - A)^{-1}B + D, \\
&= D + \frac{1}{d(s)} C \operatorname{Adj}(sI - A)B, \\
&= \frac{C \operatorname{Adj}(sI - A)B + Dd(s)}{d(s)} = \frac{N(s)}{d(s)},
\end{aligned}
$$

where $d(s) = \det(sI - A)$ is the *characteristic polynomial* of A.

2.2 Poles

Roots of the characteristic polynomial $d(s) = \det(sI - A) = 0$.

2.3 Zeros

For SISO $N(s) = [C \operatorname{Adj}(sI - A)B + Dd(s)]$ is a polynomial.
$\Rightarrow$ zeros are the roots of $N(s)$.

For MISO $N(s) = C \operatorname{Adj}(sI - A)B + Dd(s)$ is a polynomial matrix.
$\Rightarrow$ we will need a better definitions of zeros for MIMO systems!

2.4 Markov parameters

Geometric series

$$\frac{1}{1-x} = 1 + x + x^2 + \ldots, \quad \text{for } |x| < 1$$

Extension

$$(I - X)^{-1} = I + X + X^2 + \ldots,$$

WARNING: More on the radius of convergence later!

Therefore

$$(sI - A)^{-1} = s^{-1}(I - s^{-1}A) = s^{-1} + s^{-2}A + s^{-3}A^2 + \ldots,$$

and

$$\begin{aligned} H(s) &= C(sI - A)^{-1}B + D, \\ &= D + C(s^{-1} + s^{-2}A + s^{-3}A^2 + \ldots)B, \\ &= D + \sum_{i=1}^{\infty} s^{-i} H_i, \quad H_i = CA^{i-1}B \end{aligned}$$

H_i are the *Markov parameters*.

2.4.1 Interpretation for discrete-time systems

$$
\begin{aligned}
x(k+1) &= Ax(k) + Bu(k), \quad x(0) = x_0 \\
y(k) &= Cx(k) + Du(k)
\end{aligned}
$$

Then

$$
\begin{aligned}
x(1) &= Ax(0) + Bu(0), \\
x(2) &= Ax(1) + Bu(1) = A^2x(0) + ABu(0) + Bu(1), \\
x(3) &= Ax(2) + Bu(2) = A^3x(0) + A^2Bu(0) + ABu(1) + Bu(2), \\
&\vdots \\
x(k) &= A^kx(0) + \sum_{i=1}^{k} A^{i-1}Bu(k-i),
\end{aligned}
$$

and

$$
\begin{aligned}
y(k) &= CA^kx(0) + \sum_{i=1}^{k} CA^{i-1}Bu(k-i) + Du(k), \\
&= CA^kx(0) + \sum_{i=1}^{k} H_i u(k-i) + Du(k),
\end{aligned}
$$

WARNING: As a side effect, we have also computed an expression for the response of a discrete-time LTI!

2.5 Coordinate transformation

LTI system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

Coordinate transformation

$$x = Tz$$

where T is nonsingular. Note that

$$\dot{x} = T\dot{z}$$

and

$$\begin{aligned} T\dot{z} &= ATz + Bu \qquad \text{(a)} \\ y &= CTz + Du \end{aligned}$$

Multiplying (a) by T^{-1} on the left

$$\begin{aligned} \dot{z} &= T^{-1}ATz + T^{-1}Bu \\ y &= CTz + Du \end{aligned} \qquad (1)$$

2.6 Invariant parameters under coordinate transformation

2.6.1 Characteristic polynomial

$$\begin{aligned} d(s) &= \det(sI - T^{-1}AT), \\ &= \det(sT^{-1}T - T^{-1}AT), \\ &= \det[T^{-1}(sI - A)T], \\ &= \det T^{-1} \det(sI - A) \det T, \\ &= \det(sI - A). \end{aligned}$$

We used $\det(AB) = \det A \det B$ and $\det A^{-1} = 1/\det A$.

2.6.2 Transfer function

$$\begin{aligned} H(s) &= CT(sI - T^{-1}AT)^{-1}T^{-1}B + D, \\ &= CT(sT^{-1}T - T^{-1}AT)^{-1}T^{-1}B + D, \\ &= CT[T^{-1}(sI - A)T]^{-1}T^{-1}B + D, \\ &= CTT^{-1}(sI - A)^{-1}TT^{-1}B + D, \\ &= C(sI - A)^{-1}B + D. \end{aligned}$$

We used $(AB)^{-1} = B^{-1}A^{-1}$.

2.6.3 Markov parameters

$$\begin{aligned} H_i &= CT(T^{-1}AT)^{i-1}T^{-1}B, \qquad i = 1, \ldots \\ &= CTT^{-1}A^{i-1}TT^{-1}B, \\ &= CA^{i-1}B. \end{aligned}$$

We used $(T^{-1}AT)^n = T^{-1}ATT^{-1}AT \ldots T^{-1}AT = T^{-1}A^nT$.