1 Realizations of differential equations (analog simulation)

Using transfer functions,

$$Y(s) = \frac{b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3} U(s) = N(s) D^{-1}(s) U(s),$$

the controller realization splits the above as

$$Y(s) = N(s) \underbrace{D^{-1}(s) U(s)}_{X(s)}$$

that is

$$Y(s) = (b_1 s^2 + b_2 s + b_3) X(s) = N(s) X(s)$$

and

$$X(s) = \frac{1}{s^3 + a_1 s^2 + a_2 s + a_3} U(s) = D(s)^{-1} U(s)$$

If we revert

$$Y(s) = D^{-1}(s) \underbrace{N(s) U(s)}_{X(s)}$$

so that

$$X(s) = (b_1 s^2 + b_2 s + b_3) U(s) = N(s) U(s)$$

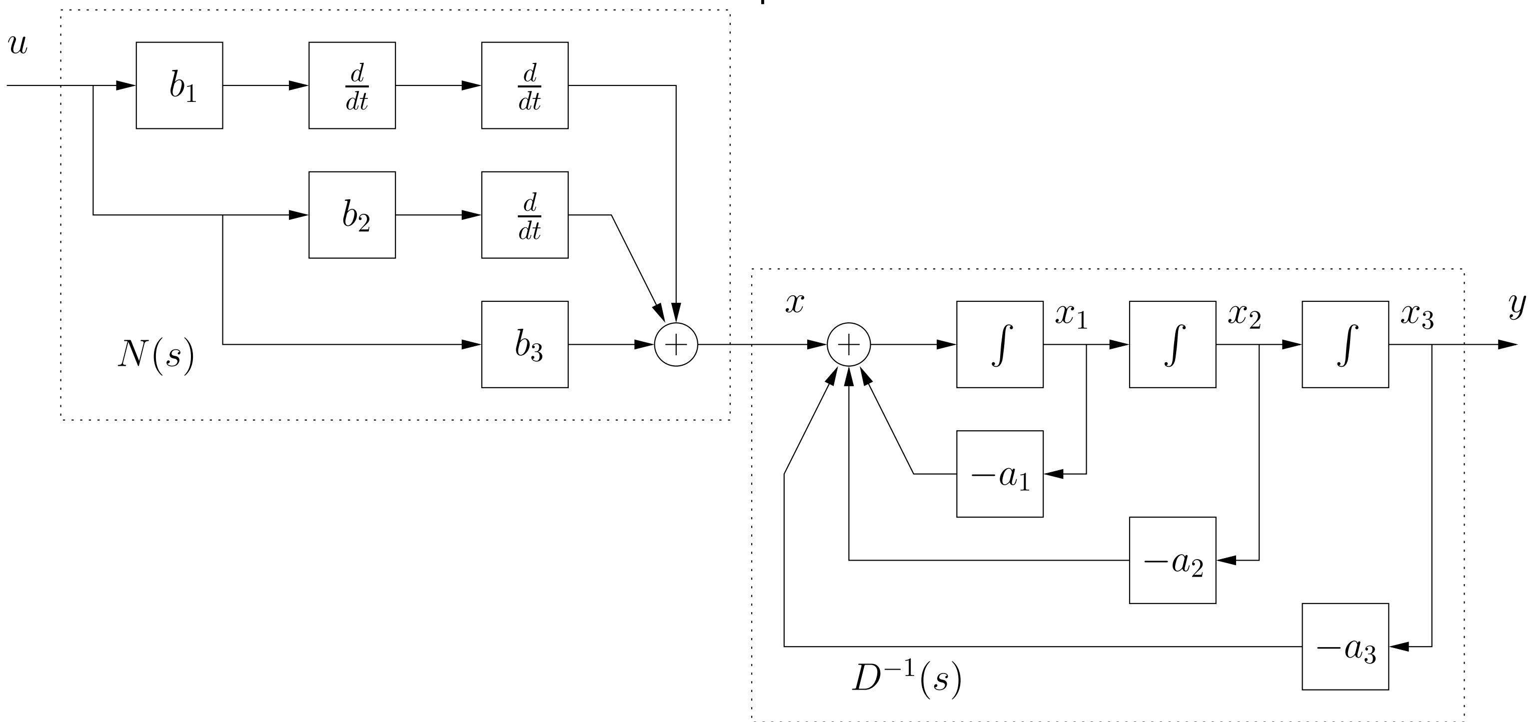
and

$$Y(s) = \frac{1}{s^3 + a_1 s^2 + a_2 s + a_3} X(s) = D(s)^{-1} X(s)$$

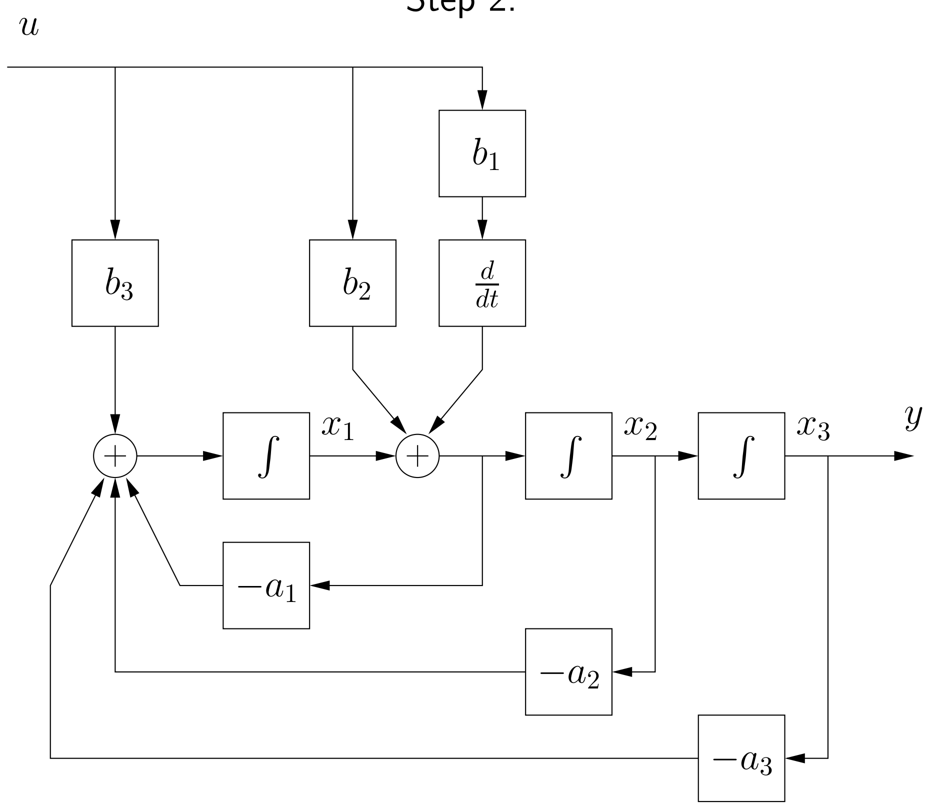
we get another realization: the observability realization.

1.1 Observability realization

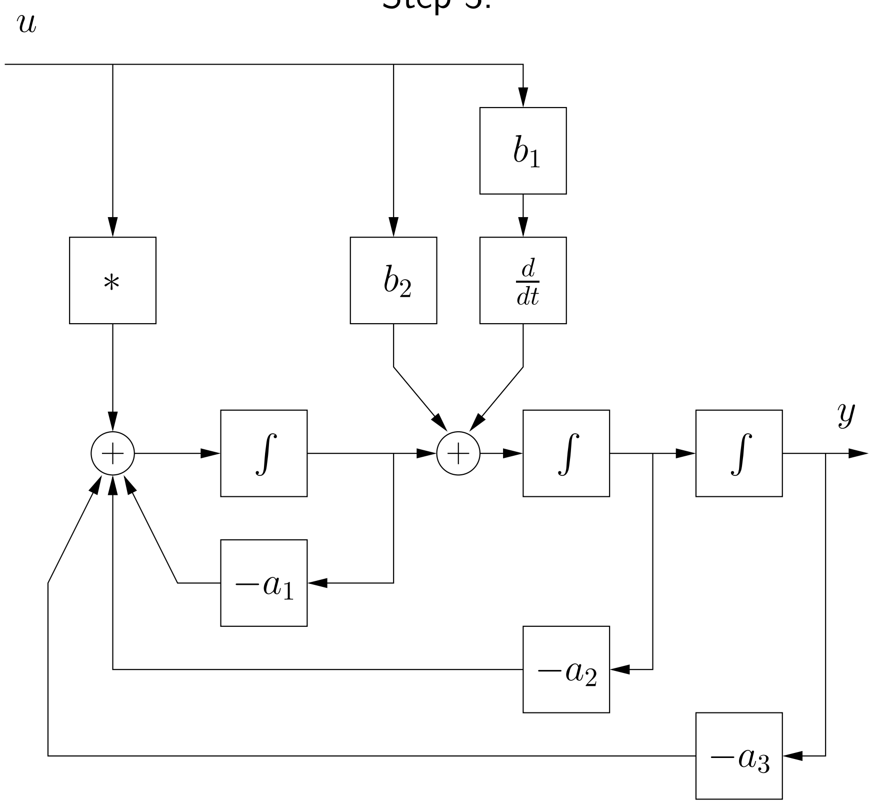
Step 1:



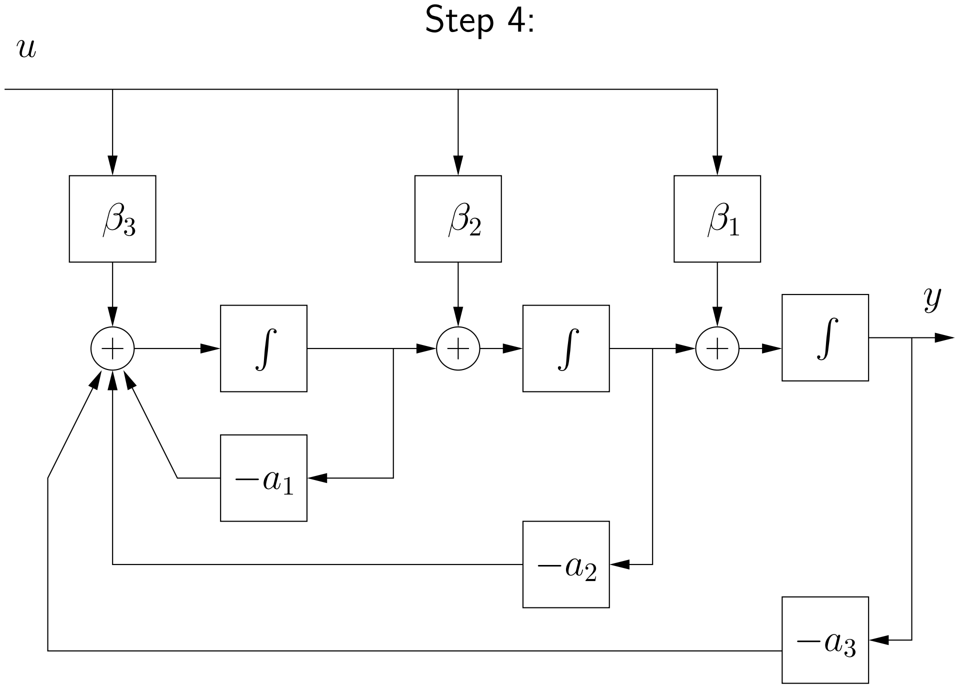
Step 2:



Step 3:



Step 4:



To compensate for feedback, manipulate the equations

$$\dot{x}_1 = -a_1 x_1 - a_2 x_2 - a_3 x_3 + b_3 u + b_2 \dot{u} + b_1 \ddot{u}, \tag{1}$$
$$\dot{x}_2 = x_1, \tag{2}$$
$$\dot{x}_3 = x_2 \tag{3}$$

First move the inputs in (1) over the integrator

$$\frac{d}{dt}(x_1 - b_2 u - b_1 \dot{u}) = -a_1 x_1 - a_2 x_2 - a_3 x_3 + b_3 u. \tag{4}$$

Now define

$$\xi = x_1 - b_2 u - b_1 \dot{u} \quad \Rightarrow \quad x_1 = \xi + b_2 u + b_1 \dot{u}, \tag{5}$$

and rewrite (4)

$$\dot{\xi} = -a_1 \xi - a_2 x_2 - a_3 x_3 + (b_3 - a_1 b_2) u - a_1 b_1 \dot{u}.$$

Play the same trick again

$$\frac{d}{dt}(\xi + a_1 b_1 u) = -a_1 \xi - a_2 x_2 - a_3 x_3 + (b_3 - a_1 b_2) u. \tag{6}$$

Define

$$\eta = \xi + a_1 b_1 u \quad \Rightarrow \quad \xi = \eta - a_1 b_1 u, \tag{7}$$

and rewrite (6)

$$\dot{\eta} = -a_1 \eta - a_2 x_2 - a_3 x_3 + (b_3 - a_1 b_2 + a_1^2 b_1) u. \tag{8}$$

Now the other equations. Use (5) and (7) to rewrite (2) as

$$\dot{x}_2 = x_1 = \eta + (b_2 - a_1 b_1) u + b_1 \dot{u}. \tag{9}$$

The same old trick...

$$\nu = x_2 - b_1 u \quad \Rightarrow \quad x_2 = \nu + b_1 u, \tag{10}$$

produces

$$\dot{\nu} = \eta + (b_2 - a_1 b_1) u. \tag{11}$$

Use (10) in (8)

$$\dot{\eta} = -a_1 \eta - a_2 \nu - a_3 x_3 + (b_3 - a_1 b_2 - a_2 b_1 + a_1^2 b_1) u \tag{12}$$

and (2)

$$\dot{x}_3 = x_2 = \nu + b_1 u. \tag{13}$$

State space equations

$$\begin{pmatrix} \dot{x}_3 \\ \dot{\nu} \\ \dot{\eta} \end{pmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} \begin{pmatrix} x_3 \\ \nu \\ \eta \end{pmatrix} + \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} u,$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{pmatrix} x_3 \\ \nu \\ \eta \end{pmatrix},$$

where

$$\begin{aligned} \beta_1 &= b_1, \\ \beta_2 &= b_2 - a_1 b_1, \\ \beta_3 &= b_3 - a_1 b_2 - a_2 b_1 + a_1^2 b_1. \end{aligned}$$

Note that

$$\begin{aligned} \beta_1 &= b_1, \\ \beta_2 &= b_2 - a_1 \beta_1, \\ \beta_3 &= b_3 - a_1 \beta_2 - a_2 \beta_1. \end{aligned}$$

or using matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ a_1 & 1 & 0 \\ a_2 & a_1 & 1 \end{bmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \qquad \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ a_1 & 1 & 0 \\ a_2 & a_1 & 1 \end{bmatrix}^{-1} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$

1.2 Observer realization

Let's propose an alternative realization to $Y(s) = D(s)^{-1}X(s)$

$$\dddot{y} + a_1\ddot{y} + a_2\dot{y} + a_3 y = x$$

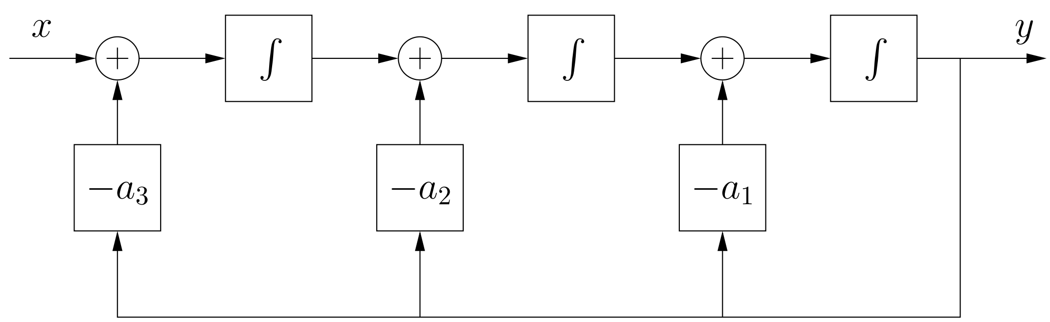
If we integrate (in time) three times

$$y + a_1 \int y + a_2 \int\int y + a_3 \int\int\int y = \int\int\int x$$

or

$$y = \int \left\{ -a_1 y + \int \left[-a_2 y + \int (x - a_3 y) \right] \right\}$$

This can be realized by the block diagram



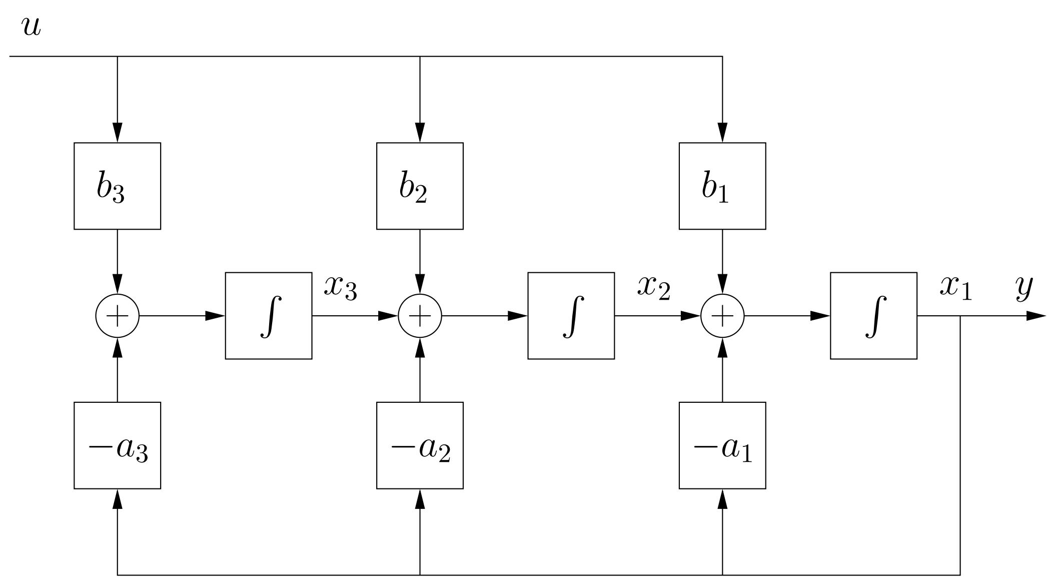
The complete equation is realized by replacing

$$x = b_1\ddot{u} + b_2\dot{u} + b_3 u$$

to get

$$y = \int \left\{ b_1 u - a_1 y + \int \left[b_2 u - a_2 y + \int (b_3 u - a_3 y) \right] \right\}$$

This can be realized by the block diagram



State space equations

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{bmatrix} -a_1 & 1 & 0 \\ -a_2 & 0 & 1 \\ -a_3 & 0 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} u,$$

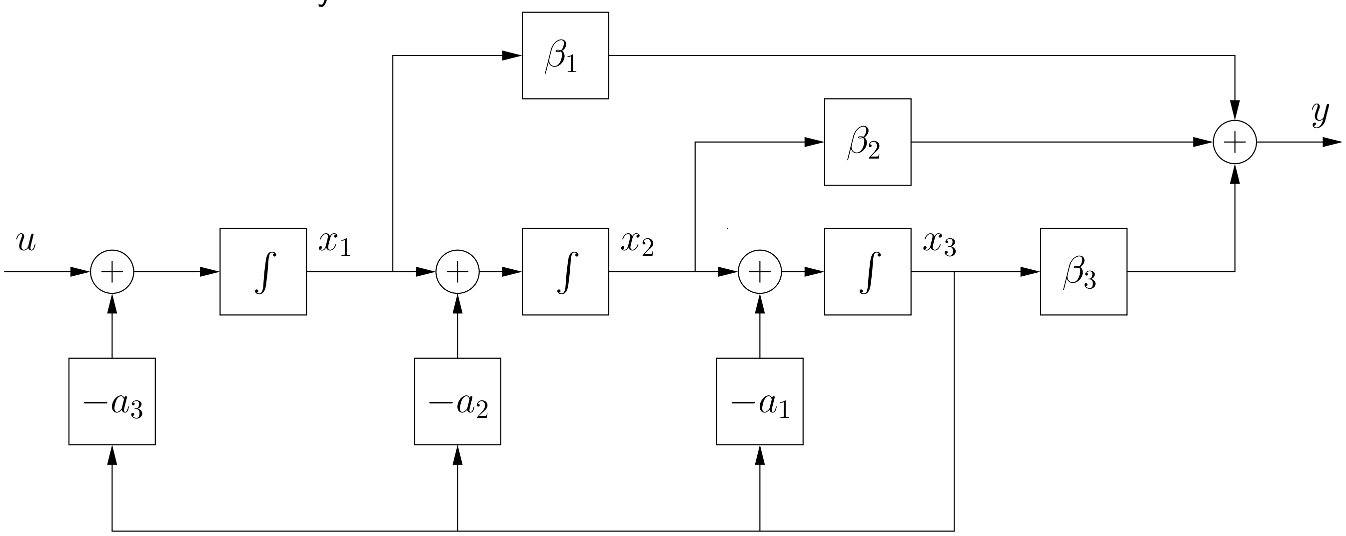
$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

1.3 Controllability realization

What do we get if we apply the alternative realization used in the observer realization to realize

$$Y(s) = N(s) \underbrace{D^{-1}(s)U(s)}_{X(s)}$$

The controllability realization



State space equations

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{bmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u,$$

$$y = \begin{bmatrix} \beta_1 & \beta_2 & \beta_3 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

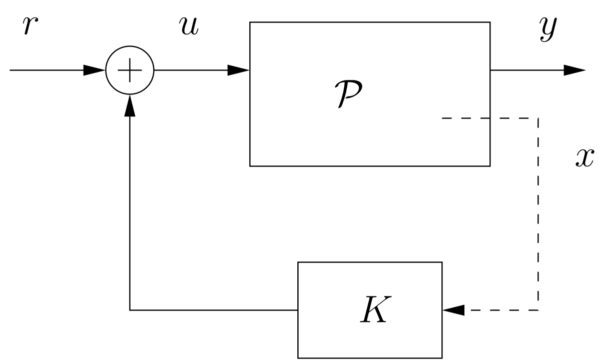
where

$$\begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ a_1 & 1 & 0 \\ a_2 & a_1 & 1 \end{bmatrix}^{-1} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

2 Summary on realizations (so far!)

- One transfer function (differential equation) admits multiple realizations.
- Distinct realizations have distinct numerical properties (computation).
- Choosing the best realization for computation is a research topic.
- Some realizations are better than others for some things.

3 Our first control problem: pole placement



State space system

$$\mathcal{P} : \begin{cases} \dot{x} = Ax + Bu, \\ y = Cx. \end{cases}$$

State feedback controller

$$u = Kx + r.$$

> **POLE PLACEMENT: Compute K such that the poles of the closed loop system are the same as the roots of the given polynomial $p(s)$.**

3.1 Closed-loop system

Substitute u into the plant

$$\begin{aligned}\dot{x} &= (A + BK)x + Br,\\ y &= Cx.\end{aligned}$$

Transfer function

$$H(s) = C(sI - A - BK)^{-1}B.$$

3.2 System poles:

Cramer's Rule

$$M^{-1} = \frac{1}{\det M} \operatorname{Adj} M,$$

where

$$\operatorname{Adj} M = [\gamma_{ij}]^T, \quad \gamma_{ij} \to \text{cofactor of } a_{ij}.$$

Application

$$(sI - M)^{-1} = \frac{1}{d(s)} N(s),$$

where

$$d(s) = \det(sI - M), \quad N(s) = \operatorname{Adj}(sI - M)$$

3.3 Pole placement

Compute K such that $\det(sI - A - BK) = p(s)$.

3.4 One particular solution

If (A, B, C) is in controller realization, say

$$\begin{aligned} A + BK &= \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}, \\ &= \begin{bmatrix} k_1 - a_1 & k_2 - a_2 & k_3 - a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \end{aligned}$$

and

$$\det(sI - A - BK) = s^3 + (a_1 - k_1)s^2 + (a_2 - k_2)s + (a_3 - k_3).$$

Hence, for $p(s) = s^3 + p_1 s^2 + p_2 s + p_3$,

$$\det(sI - A - BK) = p(s)$$

when

$$a_i - k_i = p_i, \qquad \Rightarrow \qquad k_i = a_i - p_i, \quad \forall i$$

3.5 General solution

Can we transform (A, B, C) into controller realization?