1 Stability

1.1 Bounded Input-Bounded Output (BIBO) Stability

Definition: A system $y = Hu$ is BIBO stable if for any bounded input $u(t)$ corresponds a bounded output $y(t)$.

In general, the input $u(t)$ and the output $y(t)$ are bounded in the sense of a signal norm! A scalar signal $u(t)$ is bounded if

$$\exists\ M_u < \infty : \|u(t)\| = \sup_{t \geq 0} |u(t)| < M_u.$$

Definition: A scalar function $h(t)$ is absolutely integrable if $\exists\ M_h < \infty$ such that

$$\int_0^\infty |h(\tau)|\, d\tau < M_h.$$

Theorem: The SISO linear system with impulse response $h(t)$ is BIBO stable if and only if $h(t)$ is absolutely integrable.

Theorem: The MIMO linear system with impulse response matrix $H(t) = (H_{ij}(t))$ is BIBO stable if and only if $h_{ij}(t)$ is absolutely integrable for all i, j.

Corollary: The MIMO linear system with a rational and proper transfer matrix $H(s) = N(s)/d(s)$ is BIBO stable if and only if all poles of $H(s)$, i.e., the roots of $d(s)$, are in the open left-half of the complex plane.

Proof: The impulse response $h_{ij}(t)$ can be obtained from $H_{ij}(s)$ as

$$h_{ij}(t) = \mathcal{L}^{-1}\left\{H_{ij}(s)\right\} = \mathcal{L}^{-1}\left\{\frac{N_{ij}(s)}{d(s)}\right\}.$$

Expanding $H_{ij}(s)$ in partial fractions we have

$$h_{ij}(t) = \mathcal{L}^{-1}\left\{\sum_{i=1}^{m}\sum_{j=1}^{k_i}\frac{\alpha_i}{(s-\lambda_i)^j}\right\} = \sum_{i=1}^{m}\sum_{j=1}^{k_i} t^{j-1}e^{\lambda_i t}$$

where λ_i denotes the ith root of $d(s)$ with multiplicity k_i. Therefore, $h_{ij}(t)$ is absolutely integrable if and only if the factors

$$t^{j-1}e^{\lambda_i t}, \quad j = 1, \ldots, k_i$$

are absolutely integrable. That is, iff $\mathbb{R}(\lambda_i) < 0$.

Proof of Theorem (SISO): *Sufficiency:* For linear systems

$$y(t) = \int_0^\infty h(\tau)u(t-\tau)\,d\tau$$

and for SISO systems

$$\begin{aligned} |y(t)| &= \left|\int_0^\infty h(\tau)u(t-\tau)\,d\tau\right|, \\ &\leq \int_0^\infty |h(\tau)u(t-\tau)|\,d\tau, \\ &\leq \int_0^\infty |h(\tau)||u(t-\tau)|\,d\tau. \end{aligned}$$

Therefore, since $u(t)$ is bounded and $h(t)$ is absolutely integrable

$$\begin{aligned} |y(t)| &\leq \int_0^\infty |h(\tau)||u(t-\tau)|\,d\tau, \\ &\leq M_u \int_0^\infty |h(\tau)|\,d\tau, \\ &\leq M_u M_h \end{aligned}$$

which implies $y(t)$ is bounded.

Necessity: If $h(t)$ is not absolutely integrable then there exists $\bar{t}$ such that

$$\int_0^{\bar{t}} |h(\tau)|\,d\tau = \infty.$$

Now for the particular input

$$u(\bar{t}-t) = \begin{cases} +1, & h(t) \geq 0 \\ -1, & h(t) < 0 \end{cases}$$

we have that

$$y(\bar{t}) = \int_0^{\bar{t}} h(\tau)u(\bar{t}-\tau)\,d\tau = \int_0^{\bar{t}} |h(\tau)|\,d\tau = \infty$$

which implies $y(t)$ is not bounded.

1.2 Internal Stability

The LTI system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t). \end{aligned} \qquad (1)$$

is BIBO stable iff $H(s) = C(sI - A)^{-1}B + D$ has all poles on the open left-half of the complex plane.

The LTI system (1) is *internally stable* iff all roots of $d(s) = \det(sI - A)$ are on the open left-half of the complex plane.

Internal stability $\Longrightarrow$ BIBO stability
Internal stability $\Longleftarrow$ BIBO stability $+$
controllability and observability

1.3 Lyapunov Stability

Consider the dynamic system

$$\dot{x} = f(x).$$

Definition: $\bar{x}$ is an equilibrium point of (1.3) if $f(\bar{x}) = 0$.

Definition: Let $\bar{x} = 0$ be an equilibrium point of (1.3) and let $\Omega \subset \mathbb{R}^n$. It is

1) *Stable* if for each $\epsilon > 0$ there is $\delta > 0$ such that

$$\|x(0)\| < \delta \quad \Rightarrow \quad \|x(t)\| < \epsilon, \quad \forall t \geq 0.$$

2) *Unstable* if not stable.

3) *Asymptotically stable* if it is stable and $\delta > 0$ can be chosen such that

$$\|x(0)\| < \delta \quad \Rightarrow \quad \lim_{t \to \infty} x(t) = 0.$$

Theorem: Let $x = 0$ be an equilibrium point of the dynamic system (1.3) and $x \in \Omega \subset \mathbb{R}^n$. Let $V : \Omega \to \mathbb{R}$ be continuously differentiable and

$$V(0) = 0, \qquad V(x) > 0 \quad \forall x \in \Omega, \quad x \neq 0,$$

and

$$\dot{V}(x) \leq 0, \forall x \in \Omega.$$

Then $x = 0$ is *stable*. Moreover, if

$$\dot{V}(x) < 0, \forall x \in \Omega, x \neq 0$$

then $x = 0$ is *asymptotically stable*.

Proof: see H. K. Kalil, *Nonlinear Systems*, Chapter 3.

1.3.1 Lyapunov Stability for Linear Systems

Consider the LTI system

$$\dot{x} = Ax. \tag{2}$$

Theorem: The following statements about the linear system (2) are equivalent:

1) $\mathbb{R}(\lambda_i(A)) < 0$.

2) $x = 0$ is the unique equilibrium point of (2) and it is asymptotically stable.

Proof: Use the same tools to conclude that

$$x(t) = \mathcal{L}\left\{(sI - A)^{-1}x(0)\right\}$$

so that

$$\lim_{t\to\infty} x(t) = 0$$

for any $x(0)$ bounded iff $\mathbb{R}(\lambda_i(A)) < 0$. This implies A is nonsingular, which implies that $Ax = 0 \Rightarrow x = 0$ is the unique equilibrium point.

Theorem: The LTI system (2) is *asymptotically stable* if and only if, for *any* matrix $Q > 0$, the *Lyapunov equation*

$$A^T P + PA + Q = 0 \tag{3}$$

has a unique solution P such that $P > 0$.

Proof: *Sufficiency:* Assume there exists $P > 0$ that solves (3). Then $V(x) = x^T P x > 0$, for all $0 \neq x \in \Omega := \mathbb{R}^n$. Note that

$$\begin{aligned}
\dot{V}(x) &= \frac{d}{dt} x^T P x, \\
&= \dot{x}^T P x + \dot{x}^T P x, \\
&= x^T (A^T P + PA) x, \\
&= -x^T Q x < 0, \quad \forall\, 0 \neq x \in \mathbb{R}^n.
\end{aligned}$$

Therefore (2) is asymptotically stable.

Necessity: Assume (2) is asymptotically stable then $\mathbb{R}(\lambda_i(A)) < 0$. This can be used to show that the Gramian

$$P = \int_0^\infty e^{A^T t} Q e^{At} dt$$

converges to a finite value. Using stability we can also show that

$$\lim_{t \to \infty} e^{A^T t} Q e^{At} = 0.$$

Therefore, as we already know

$$\begin{aligned}
A^T P + PA &= A^T \left(\int_0^\infty \frac{d}{dt} e^{A^T t} Q e^{At} dt \right) + \left(\int_0^\infty \frac{d}{dt} e^{A^T t} Q e^{At} dt \right) A, \\
&= \int_0^\infty \frac{d}{dt} e^{A^T t} Q e^{At} dt, \\
&= \lim_{t \to \infty} (e^{A^T t} Q e^{At}) - Q = -Q.
\end{aligned}$$

To prove that it is positive definite we assume that it is not, so that there exists $z \neq 0$ such that (see a similar proof for the Controllability and Observability Gramians)

$$z^* P = z^* \int_0^\infty e^{A^T t} Q e^{At} dt = 0 \quad \Rightarrow \quad e^{At} z \equiv 0, \ \forall\, t \geq 0 \quad \Rightarrow \quad z = 0.$$

Finally, to prove that it is unique assume there exists $\tilde{P} \neq P$ satisfying the Lyapunov equation. Then

$$A^T(P - \tilde{P}) + (P - \tilde{P})A = 0$$

and

$$\begin{aligned} e^{A^T t}\left[A^T(P - \tilde{P}) + (P - \tilde{P})A\right]e^{At} &= A^T\left[e^{A^T t}(P - \tilde{P})e^{At}\right] + \left[e^{A^T t}(P - \tilde{P})e^{At}\right]A, \\ &= \frac{d}{dt}e^{A^T t}(P - \tilde{P})e^{At} = 0, \quad \forall t \geq 0, \end{aligned}$$

which implies that

$$e^{A^T t}(P - \tilde{P})e^{At} \text{ is constant} \quad \forall t \geq 0.$$

In particular, at $t = 0$ and $t \to \infty$

$$(P - \tilde{P}) = \lim_{t \to \infty} e^{A^T t}(P - \tilde{P})e^{At} = 0 \quad \Rightarrow \quad \tilde{P} = P$$

REMARK #1: The proof works also if $Q = C^T C \geq 0$ and (A, C) is observable.

REMARK #2: Internal stability $\iff$ Asymptotic stability.

1.4 Example

$$\dot{x} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} x$$

Lyapunov Equation ($Q = I$)

$$\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}^T \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} = - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

is equivalent to the linear system

$$\begin{bmatrix} 0 & 2 & 0 \\ -1 & -1 & 1 \\ 0 & -2 & -2 \end{bmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}$$

which has the unique solution

$$\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} 3/2 \\ -1/2 \\ 1 \end{pmatrix} \quad \Rightarrow \quad \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} = \begin{bmatrix} 3/2 & -1/2 \\ -1/2 & 1 \end{bmatrix} > 0$$