1 Minimal Realizations

Definition: A realization (A, B, C) is *minimal* if (A, B) is controllable and (A, C) is observable.

Theorem: For any (A, B, C) there exists a nonsingular similarity transformation T that transforms the original system into the form

$$\bar{A} = T^{-1}AT = \begin{bmatrix} A_{co} & 0 & A_o & 0 \\ A_c & A_{c\bar{o}} & A_1 & A_2 \\ 0 & 0 & A_{\bar{c}o} & 0 \\ 0 & 0 & A_3 & A_{\bar{c}\bar{o}} \end{bmatrix}, \qquad \bar{B} = T^{-1}B = \begin{bmatrix} B_{co} \\ B_{c\bar{o}} \\ 0 \\ 0 \end{bmatrix},$$

$$\bar{C} = CT = \begin{bmatrix} C_{co} & 0 & C_{\bar{c}o} & 0 \end{bmatrix}.$$

In this form:

1) The subsystem (A_{co}, B_{co}, C_{co}) is controllable and observable and $C(sI - A)^{-1}B = C_{co}(sI - A_{co})^{-1}B_{co}$;

2) The subsystem
$$\left(\begin{bmatrix} A_{co} & 0 \\ A_c & A_{c\bar{o}} \end{bmatrix}, \begin{bmatrix} B_{co} \\ B_{c\bar{o}} \end{bmatrix}, \begin{bmatrix} C_{co} & 0 \end{bmatrix} \right)$$
is controllable;

3) The subsystem
$$\left(\begin{bmatrix} A_{co} & A_o \\ 0 & A_{\bar{c}o} \end{bmatrix}, \begin{bmatrix} B_{co} \\ 0 \end{bmatrix}, \begin{bmatrix} C_{co} & C_{\bar{c}o} \end{bmatrix} \right)$$
is observable;

4) The subsystem $(A_{\bar{c}\bar{o}}, 0, 0)$ is neither controllable nor observable.

Problem: Given a non-minimal realization (A, B, C, D) find an equivalent realization $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ that is minimal.

2 The Singular Value Decomposition

Theorem: For any matrix $A \in \mathbb{R}^{m\times n}$ with $\operatorname{rank}(A) = r < n$ there exist real square matrices

$$U = \begin{bmatrix} U_1 & U_2 \end{bmatrix}, \qquad V = \begin{bmatrix} V_1 & V_2 \end{bmatrix},$$

and a diagonal real matrix

$$\Sigma = \begin{bmatrix} \Sigma_+ & 0 \\ 0 & 0 \end{bmatrix},$$

such that

$$A = U\Sigma V^T = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_+ & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix},$$

where

$$UU^T = U_1U_1^T + U_2U_2^T = I, \qquad VV^T = V_1V_1^T + V_2V_2^T = I,$$

and $\Sigma_+ > 0$.

Proof: There exists orthogonal U and V such that

$$AA^T = U\Lambda_1U^T, \qquad A^TA = V\Lambda_2V^T,$$

where $\Lambda_1 \geq 0$, $\Lambda_2 \geq 0$. Also, if $A = U\Sigma V$ then

$$AA^T = U\Sigma V^T V\Sigma^T U^T = U\Sigma\Sigma^T U^T,$$
$$A^TA = V\Sigma^T U^T U\Sigma V^T = V\Sigma^T\Sigma V^T.$$

One can prove that the choice

$$\Lambda_1 = \Sigma\Sigma^T, \qquad \Lambda_2 = \Sigma^T\Sigma,$$

is possible.

3 Computing Minimal Realizations

3.1 Computing Controllable Realizations

Let (A, B, C) be a non-minimal realization.

1) Compute $\mathcal{C}(A, B)$.

2) Compute the singular value decomposition

$$\mathcal{C}(A, B) = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_+ & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} = U_1 \Sigma_+ V_1^T,$$

where $\Sigma_+ > 0$ and

$$U_1 U_1^T + U_2 U_2^T = I, \qquad V_1 V_1^T + V_2 V_2^T = I.$$

3) Set

$$r = \operatorname{rank} \mathcal{C}(A, B) = \dim \Sigma_+.$$

4) Compute

$$R_1 = U_1 \Sigma_+^{-1/2} \qquad T_1 = U_1 \Sigma_+^{1/2}.$$

Theorem: The realization

$$(\bar{A}, \bar{B}, \bar{C}) = \left(R_1^T A T_1, R_1^T B, C T_1 \right)$$

of order r is controllable and $B(sI - A)^{-1} C = \bar{B}(sI - \bar{A})^{-1} \bar{C}$.

Proof: First note that

$$U^T U = \begin{bmatrix} U_1^T \\ U_2^T \end{bmatrix} \begin{bmatrix} U_1 & U_2 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \quad \Rightarrow \quad U_1^T U_1 = I,$$

so that for

$$P = P^T = R_1 T_1^T = U_1 \Sigma_+^{-1/2} \Sigma_+^{1/2} U_1^T = U_1 U_1^T$$

we have

$$P\mathcal{C}(A, B) = (U_1 U_1^T)(U_1 \Sigma_+ V_1^T) = U_1 \Sigma_+ V_1^T = \mathcal{C}(A, B).$$

That is

$$PC(A,B) = \begin{bmatrix} PB & PAB & \cdots & PA^{n-1}B \end{bmatrix} = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}.$$

Therefore

$$\begin{aligned} (AP)B &= A(PB) = AB, \\ (AP)^2B = AP(APB) = AP(AB) &= A(PAB) = A^2B, \\ &\vdots \\ (AP)^nB = APA^{n-1}B &= A^nB. \end{aligned}$$

Consequently

$$\begin{aligned} \bar{A}^i\bar{B} &= (R_1^T A T_1)^i R_1^T B, \\ &= (R_1^T A T_1)(R_1^T A T_1)\cdots(R_1^T A T_1)R_1^T B, \\ &= R_1^T (A T_1 R_1^T)(A T_1 R_1^T)\cdots(A T_1 R_1^T)B, \\ &= R_1^T (AP)^i B, \\ &= R_1^T A^i B, \end{aligned}$$

so that

$$\begin{aligned} \mathcal{C}(\bar{A},\bar{B}) &= \begin{bmatrix} \bar{B} & \bar{A}\bar{B} & \cdots & \bar{A}^{n-1}\bar{B} \end{bmatrix}, \\ &= R_1^T \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}, \\ &= R_1^T \mathcal{C}(A,B) \end{aligned}$$

As R_1 has full-column, $\operatorname{rank}\mathcal{C}(\bar{A},\bar{B}) = r$ and therefore $(\bar{A},\bar{B})$ is controllable. Finally remember that

$$C\,(sI - A)^{-1}\,B = \sum_{i=1}^{\infty} H_i, \quad H_i = CA^{i-1}B.$$

Since

$$\begin{aligned} \bar{H}_i &= \bar{C}(\bar{A}^{i-1}\bar{B}), \\ &= (CT_1)(R_1^T A^{i-1} B), \\ &= C(PA^{i-1}B), \\ &= C(A^{i-1}B) = H_i, \end{aligned}$$

we conclude that

$$C\,(sI - A)^{-1}\,B = \bar{C}\left(sI - \bar{A}\right)^{-1}\bar{B}.$$

3.2 Computing Observable Realizations

Let (A, B, C) be a non-minimal realization.

1) Compute $\mathcal{O}(A, C)$.

2) Compute the singular value decomposition

$$\mathcal{O}(A, C) = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_+ & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} = U_1 \Sigma_+ V_1^T,$$

where $\Sigma_+ > 0$ and

$$U_1 U_1^T + U_2 U_2^T = I, \qquad V_1 V_1^T + V_2 V_2^T = I.$$

3) Set

$$r = \operatorname{rank} \mathcal{O}(A, C) = \dim \Sigma_+.$$

4) Compute

$$R_1 = V_1 \Sigma_+^{1/2} \qquad T_1 = V_1 \Sigma_+^{-1/2}.$$

Theorem: The realization

$$(\bar{A}, \bar{B}, \bar{C}) = \left(R_1^T A T_1, R_1^T B, C T_1 \right)$$

of order r is observable and $B(sI - A)^{-1} C = \bar{B}(sI - \bar{A})^{-1} \bar{C}$.

Proof: Note that

$$P = P^T = R_1 T_1^T = V_1 \Sigma_+^{1/2} \Sigma_+^{-1/2} V_1^T = V_1 V_1^T$$

and

$$P \mathcal{O}(A, C)^T = (V_1 V_1^T)(U_1 \Sigma_+ V_1^T)^T = \mathcal{O}(A, C)^T$$

and follow the same steps as in the previous proof.

3.3 Computing Minimal Realizations

Let (A, B, C) be a non-minimal realization.

1) Compute $\mathcal{C}(A, B)$ and $\mathcal{O}(A, C)$.

2) Compute the singular value decomposition

$$\mathcal{C}(A, B) = \begin{bmatrix} U_c & U_{\bar{c}} \end{bmatrix} \begin{bmatrix} \Sigma_c & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_c^T \\ V_{\bar{c}}^T \end{bmatrix} = U_c \Sigma_+ V_c^T,$$

where $\Sigma_c > 0$ and

$$U_c U_c^T + U_{\bar{c}} U_{\bar{c}}^T = I, \qquad V_c V_c^T + V_{\bar{c}} V_{\bar{c}}^T = I.$$

3) Compute the singular value decomposition

$$\mathcal{O}(A, C) U_c \Sigma^{1/2} = \begin{bmatrix} U_{co} & U_{\bar{c}o} \end{bmatrix} \begin{bmatrix} \Sigma_{co} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_{co}^T \\ V_{\bar{c}o}^T \end{bmatrix} = U_{co} \Sigma_+ V_{co}^T,$$

where $\Sigma_{co} > 0$ and

$$U_{co} U_{co}^T + U_{\bar{c}o} U_{\bar{c}o}^T = I, \qquad V_{co} V_{co}^T + V_{\bar{c}o} V_{\bar{c}o}^T = I.$$

4) Set

$$r = \operatorname{rank} \mathcal{O}(A, C) T_c = \dim \Sigma_{co}.$$

5) Compute

$$R_1 = U_c \Sigma_c^{-1/2} V_{co} \Sigma_{co}^{1/2} \qquad T_1 = U_c \Sigma_c^{1/2} V_{co} \Sigma_{co}^{-1/2}.$$

Theorem: The realization

$$(\bar{A}, \bar{B}, \bar{C}) = \left(R_1^T A T_1, R_1^T B, C T_1 \right)$$

of order r is a minimal realization.

Proof: Combine the two previous theorems.