1 Output Controllability

LTI system in state space

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) \end{aligned}$$

Problem: Given $x(0) = 0$ and *any* $\bar{x}$, compute $u(t)$ such that $y(\bar{t}) = \bar{y}$ for some $\bar{t} > 0$.

Solution: We know that

$$\bar{y} = y(\bar{t}) = \int_0^{\bar{t}} Ce^{A(\bar{t}-\tau)} Bu(\tau) d\tau.$$

If we limit our search for solutions u in the form

$$u(t) = B^T e^{A^T(\bar{t}-t)} C^T \bar{z}$$

we have

$$\begin{aligned} \bar{y} &= \int_0^{\bar{t}} Ce^{A(\bar{t}-\tau)} BB^T e^{A^T(\bar{t}-\tau)} C^T \bar{z} d\tau, \\ &= CX(t)C^T \bar{z}, \qquad X(t) := \int_0^t e^{A\xi} BB^T e^{A^T \xi} d\xi \end{aligned}$$

and

$$\bar{z} = \left(CX(t)C^T\right)^{-1} \bar{x}, \qquad \Rightarrow \quad u(t) = B^T e^{A^T(\bar{t}-t)} C^T \left(CX(t)C^T\right)^{-1} \bar{y}$$

As before, singularity of $CX(t)C^T$ is equivalent to the existance of

$$z \neq 0, \quad z^* Ce^{A\tau} B \equiv 0, \quad \forall\, 0 \leq \tau \leq t.$$

which implies

$$\frac{d^i}{d\tau^i}(i!\, z^* Ce^{A\tau} B)\Big|_{\tau=0} = z^* CA^i e^{A\tau} B\big|_{\tau=0} = z^* CA^i B = 0, \quad i = 0, \ldots, n-1.$$

This is equivalent to the matrix

$$\begin{bmatrix} CB & CAB & CA^2B & \ldots & CA^{n-1}B \end{bmatrix} = C\mathcal{C}(A, B)$$

having full-row rank.

1.1 Summary on Output Controllability

Theorem: The following are equivalent

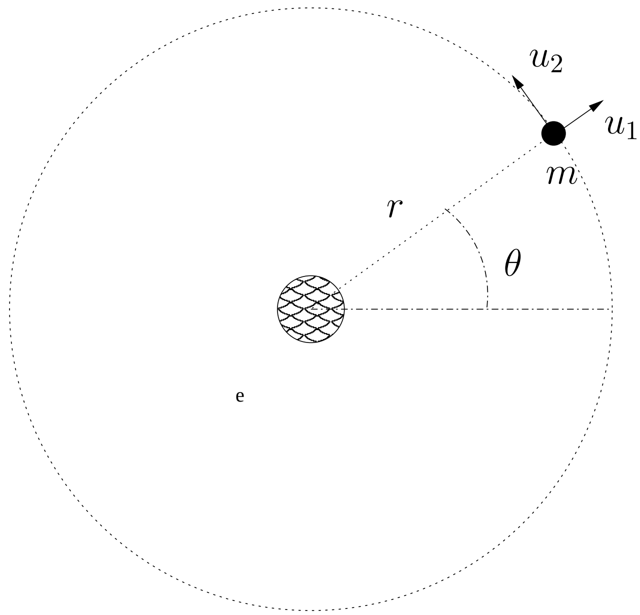
1) The triplet (A, B, C) is output controllable;

2) The matrix $C\mathcal{C}(A, B)$ has full-row rank;

3) The matrix $CX(t)C^T$ is positive definite for some $t \geq 0$.

WARNING: If C has full-row rank $X(t) > 0 \quad \Rightarrow \quad CX(t)C^T > 0$. But there might be cases when $CX(t)C^T > 0$ and $X(t) \geq 0$!

2 A complete example: satelite in circular orbit



Satelite of mass m with thrust in the radial direction u_1 and in the tangential direction u_2. From Skelton, DSC, p. 101.

Newton's law

$$m\ddot{\vec{r}} = \vec{u}_1 + \vec{u}_2 + \vec{f}_g,$$

where $\vec{f}_g$ is the gravitational force

$$\vec{f}_g = -\frac{km}{r^2}\frac{\vec{r}}{r}.$$

Using cylindrical coordinates

$$\vec{e}_1 = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \qquad \vec{e}_2 = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix},$$

we have

$$\vec{r} = r\vec{e}_1, \qquad \vec{u}_1 = u_1\vec{e}_1, \qquad \vec{u}_2 = u_2\vec{e}_2, \qquad \vec{f}_g = -\frac{km}{r^2}\vec{e}_1.$$

We need to compute

$$\ddot{\vec{r}} = \frac{d^2}{dt^2}(r\vec{e}_1) = \frac{d}{dt}(\dot{r}\vec{e}_1 + r\dot{\vec{e}}_1) = \ddot{r}\vec{e}_1 + 2\dot{r}\dot{\vec{e}}_1 + r\ddot{\vec{e}}_1,$$

where

$$\begin{aligned}
\dot{\vec{e}}_1 &= \dot{\theta}\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} = \dot{\theta}\vec{e}_2, \\
\ddot{\vec{e}}_1 &= \ddot{\theta}\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} + \dot{\theta}^2\begin{pmatrix} -\cos\theta \\ -\sin\theta \end{pmatrix} = \ddot{\theta}\vec{e}_2 - \dot{\theta}^2\vec{e}_1.
\end{aligned}$$

That is

$$\ddot{\vec{r}} = (\ddot{r} - r\dot{\theta}^2)\vec{e}_1 + (2\dot{r}\dot{\theta} + r\ddot{\theta})\vec{e}_2,$$

so that Newton's law can be rewritten as

$$m(\ddot{r} - r\dot{\theta}^2)\vec{e}_1 + m(2\dot{r}\dot{\theta} + r\ddot{\theta})\vec{e}_2 = u_1\vec{e}_1 + u_2\vec{u}_2 - \frac{km}{r^2}\vec{e}_1,$$

or, equivalently, as the two scalar differential equations

$$\begin{aligned}
m(\ddot{r} - r\dot{\theta}^2) &= u_1 - \frac{km}{r^2}, \\
m(2\dot{r}\dot{\theta} + r\ddot{\theta}) &= u_2.
\end{aligned}$$

In state space

$$x = \begin{pmatrix} r \\ \theta \\ \dot{r} \\ \dot{\theta} \end{pmatrix}, \qquad \dot{x} = \begin{pmatrix} \dot{r} \\ \dot{\theta} \\ \ddot{r} \\ \ddot{\theta} \end{pmatrix} = \begin{pmatrix} x_3 \\ x_4 \\ x_1x_4^2 - k/x_1^2 \\ -2x_3x_4/x_1 \end{pmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1/m & 0 \\ 0 & 1/(mx_1) \end{bmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

This is a *nonlinear system* and we look for equilibrium ($\ddot{r} = \ddot{\theta} = 0$) when $u_1 = u_2 = 0$. This can be stated as

$$x_1x_4^2 - k/x_1^2 = 0, \qquad\qquad -2x_3x_4/x_1 = 0.$$

The second condition implies $x_3 = \dot{r}$ and/or $x_4 = \dot{\theta}$ must be zero. We choose $x_3 = \dot{r} = 0$ which implies $x_1 = r = \bar{r}$ constant and

$$x_4 = \dot{\theta} = \sqrt{\frac{k}{x_1^3}} = \sqrt{\frac{k}{\bar{r}^3}} = \bar{\omega} \quad \Rightarrow \quad k = \bar{r}^3\bar{\omega}^2.$$

Note also that $x_2 = \theta = \bar{\omega}t$.

This nonlinear system is in the form

$$\dot{x} = f(x,t) + g(x)u.$$

We will linearize $f(x,t)$ and $g(x)u$ around the equilibrium point $(\bar{x}, \bar{u})$ to obtain the linearized system

$$\dot{x} = (\nabla f_x)^T[x(t) - \bar{x}(t)] + (\nabla g_x)^T[x(t) - \bar{x}(t)]\bar{u} + g(\bar{x})u.$$

For this problem

$$\bar{x}(t) = \begin{pmatrix} \bar{r} \\ \bar{\omega}t \\ 0 \\ \bar{\omega} \end{pmatrix}, \quad \bar{u} = 0, \quad f(x,t) = \begin{pmatrix} x_3 \\ x_4 \\ x_1 x_4^2 - k/x_1^2 \\ -2x_3x_4/x_1 \end{pmatrix}, \quad g(x) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1/m & 0 \\ 0 & 1/(mx_1) \end{bmatrix},$$

and

$$(\nabla f_x)^T = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2k/\bar{r}^3 + \bar{\omega}^2 & 0 & 0 & 2\bar{r}\bar{\omega} \\ 0 & 0 & -2\bar{\omega}/\bar{r} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 3\bar{\omega}^2 & 0 & 0 & 2\bar{r}\bar{\omega} \\ 0 & 0 & -2\bar{\omega}/\bar{r} & 0 \end{bmatrix}.$$

This produces the linearized system

$$\dot{x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 3\bar{\omega}^2 & 0 & 0 & 2\bar{r}\bar{\omega} \\ 0 & 0 & -2\bar{\omega}/\bar{r} & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1/m & 0 \\ 0 & 1/(m\bar{r}) \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

If we looking at the satelite (from the earth) we can say that we can observe r and $\dot{\theta}$ (how?), that is

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x.$$

Questions:

1) Can we know the 'state' of the satelite is by measuring only r?

2) Can we know the 'state' of the satelite is by measuring only $\dot{\theta}$?

3) Can we know the 'state' of the satelite by measuring r and $\dot{\theta}$?

4) Can the system be controlled to remain in circular orbit using radial thrusting (u_1) alone?

5) Can the system be controlled using tangential thrusting (u_2) alone?

Question: Can we know the 'state' of the satelite by measuring only r?

Answer: Is the system observable when $C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$? Compute the observability matrix

$$\mathcal{O}(A, C) = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix},$$
$$= \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ \hline 3\bar{\omega}^2 & 0 & 0 & 2\bar{r}\bar{\omega} \\ \hline 0 & 0 & -\bar{\omega}^2 & 0 \end{array}\right]$$

Phsysical interpretation: measuring r does not give any information on θ or $\bar{\theta}$!

Note that if we that know the satelite is in equilibrium and "measure" k then

$$\dot{\theta} = \bar{\omega} = \sqrt{\frac{k}{\bar{r}^3}}.$$

But we still do not know θ since

$$\theta(t) = \theta(0) + \omega t,$$

and we do not know $\theta(0)$!

Question: Can we know the 'state' of the satelite by observing $\dot{\theta}$ only?

Answer: Is the system observable when $C = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$? Compute the observability matrix

$$
\begin{aligned}
\mathcal{O}(A, C) &= \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix}, \\
&= \left[\begin{array}{cccc} 0 & 0 & 0 & 1 \\ \hline 0 & 0 & -2\bar{\omega}/\bar{r} & 0 \\ \hline -6\bar{\omega}^3/\bar{r} & 0 & 0 & -4\bar{\omega}^2 \\ \hline 0 & 0 & 2\bar{\omega}^3/\bar{r} & 0 \end{array}\right]
\end{aligned}
$$

Phsyical interpretation: again, if we try to reconstruct θ from $\dot{\theta}$ we still need to know θ at some $\bar{t}$! From that point on

$$\theta = \theta(\bar{t}) + \int_{\bar{t}}^{t} \dot{\theta}(\tau) d\tau.$$

Question: Can we know the 'state' of the satelite by measuring r and $\dot{\theta}$?

Answer: Is the system observable when $C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$? Compute the observability matrix

$$
\begin{aligned}
\mathcal{O}(A, C) &= \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix}, \\
&= \left[\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
\hline
0 & 0 & 1 & 0 \\
0 & 0 & -2\bar{\omega}/\bar{r} & 0 \\
\hline
3\bar{\omega}^2 & 0 & 0 & 2\bar{r}\bar{\omega} \\
-6\bar{\omega}^3/\bar{r} & 0 & 0 & -4\bar{\omega}^2 \\
\hline
0 & 0 & -\bar{\omega}^2 & 0 \\
0 & 0 & 2\bar{\omega}^3/\bar{r} & 0
\end{array} \right]
\end{aligned}
$$

Phsysical interpretation: can we know θ at all?

Question: Can the system be controlled to remain in circular orbit using radial thrusting (u_1) alone?

Answer: Is the system controllable when $B = \begin{bmatrix} 0 \\ 0 \\ 1/m \\ 0 \end{bmatrix}$? Compute the controllability matrix

$$
\begin{aligned}
\mathcal{O}(A, C) &= \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix}, \\
&= \frac{1}{m} \left[\begin{array}{c|c|c|c} 0 & 1 & 0 & -\bar{\omega}^2 \\ 0 & 0 & -2\bar{\omega}/\bar{r} & 0 \\ 1 & 0 & -\bar{\omega}^2 & 0 \\ 0 & -2\bar{\omega}/\bar{r} & 0 & 2\bar{\omega}^3/\bar{r} \end{array} \right]
\end{aligned}
$$

Note that

$$
-\bar{\omega}^2 \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2\bar{\omega}/\bar{r} \end{pmatrix} = \begin{pmatrix} -\bar{\omega}^2 \\ 0 \\ 0 \\ 2\bar{\omega}^3/\bar{r} \end{pmatrix}
$$

which implies that the system is not controllable from u_1!
Phsyical interpretation: there must be a change in the angular velocity $\dot{\theta}$ if one changes the radius!

Question: Can the system be controlled using tangential thrusting (u_2) alone?

Answer: Is the system controllable when $B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/(m\bar{r}) \end{bmatrix}$? Compute the controllability matrix

$$\begin{aligned} \mathcal{O}(A, C) &= \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix}, \\ &= \frac{1}{m\bar{r}} \left[\begin{array}{c|c|c|c} 0 & 0 & 2\bar{r}\bar{\omega} & 0 \\ 0 & 1 & 0 & -4\bar{\omega}^2 \\ 0 & 2\bar{r}\bar{\omega} & 0 & -2\bar{r}\bar{\omega}^3 \\ 1 & 0 & -4\bar{\omega}^2 & 0 \end{array} \right] \end{aligned}$$

The above matrix has full rank, so the system is controllable from u_2! Phsysical interpretation: we can change the radius by changing the angular velocity!