Linear Systems Theory — MAE 280A

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Bibliography

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2. Robert E. Skelton, T. Iwasaki and K. Grigoriadis, *A Unified Algebraic Approach to Control Design*, Taylor & Francis, 1997.

Additional Bibliography

On systems & control:

1. Thomas Kailath, *Linear Systems*, Prentice Hall, 1980.
2. David G. Luenberger, *Introduction to dynamic systems*, John Wiley & Sons, 1979.

On linear algebra and matrices:

1. Richard Belman, *Introduction to Matrix Analysis*, McGraw-Hill, 1960.
2. Roger A. Horn and Charles R. Johnson, *Matrix Analysis*, Cambridge University Press, 1985.
3. F. R. Gantmacher, *Matrix Theory*, Chelsea Publishing, 1959.

1 Nomenclature

1.1 System

A box with inputs (u), outputs (y), and state (x).

1.2 Continuous-time

Differential equation: $\ddot{y}(t) + 2\dot{y}(t) - y(t) = u(t)$.
Independent variable $t \in \mathbb{R}$.
State: $x(t) = (y(t), \dot{y}(t))$.

1.3 Discrete-time

Difference equation: $y(k+2) + 2y(k+1) - y(k) = u(k)$.
Independent variable $k \in \mathbb{Z}$.
State: $x(k) = (y(k), y(k+1))$.

1.4 Memoryless

$y(t)$ depends only on $u(t)$.
Example: purely resistive electric circuits.

1.5 Dynamic

$y(t)$ depends on $u(\bar{t})$, $\bar{t} \neq t$.

1.6 Causal

$y(t)$ depends on $u(\bar{t})$, $\bar{t} \leq t$.

1.7 Linear

Axiomatic definition: $y(\alpha u_1 + \beta u_2) = \alpha y(u_1) + \beta y(u_2)$.
May be deceiving! Check Kailath Chapter 1.

1.8 Non-Linear

All others :)

2 Linear Systems

2.1 SISO

Single-Input-Single-Output
Several special results
"Classic" control theory (until the 60's)

2.2 MIMO

Multiple-Input-Multiple-Output
"Modern" control theory (from the 60's on)

2.3 Time-invariant

$u(t) \to y(t) \quad \Rightarrow \quad u(t-\tau) \to y(t-\tau)$ for any τ.

2.4 Impulse

Pulse:

$$\delta_\Delta(t-\tau) = \begin{cases} 0, & t < \tau, \\ 1/\Delta, & \tau \le t \le \tau + \Delta \\ 0, & t > \tau \end{cases}$$

Dirac's Delta:

$$\delta(t-\tau) = \lim_{\Delta \to 0} \delta_\Delta(t-\tau)$$

Properties $(a < \tau < b)$,

$$\int_a^b \delta(t-\tau)dt = 1$$

$$\int_a^b f(t)\delta(t-\tau)dt = f(\tau)$$

2.5 Linear systems and impulse response

Impulse Response:

$$u(t) = \delta(t-\tau), x(0) = 0 \to y(t) = h(t,\tau).$$

Linear Time-Invariant (LTI):

$$h(t,\tau) = h(t+T, \tau+T).$$

For $T = -\tau$

$$h(t,\tau) = h(t-\tau, 0) = h(t-\tau).$$

LTI and causal:

$$h(t) = 0, \quad \forall t < 0.$$

2.6 Convolution

For any $u(t)$

$$y(t) = \int_0^t h(t-\tau)u(\tau)d\tau = \int_0^t h(t)u(t-\tau)d\tau.$$

A LTI system *is* its impulse response!

2.7 Laplace Transform

Assume we all know (check the references!)

$$f(t), t \in \mathbb{R} \stackrel{\mathcal{L}}{\longleftrightarrow} F(s), s \in \mathbb{C}$$

Laplace transform (unilateral):

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt, \quad \forall s : \mathrm{Re}(s) \geq a.$$

Inverse Laplace transform:

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{j2\pi}\int_{b-j\infty}^{b+j\infty} F(s)e^{st}ds, \quad \text{for any } b > a.$$

Properties we care:

$$\begin{aligned}\mathcal{L}\{\alpha f_1(t) + \beta f_2(t)\} &= \alpha F_1(s) + \beta F_2(s),\\ \mathcal{L}\{\frac{d}{dt}f(t)\} &= sF(s) - f(0),\\ \mathcal{L}\{e^{-at}f(t)\} &= sF(s-a).\end{aligned}$$

Some pairs

$$\begin{aligned}\delta(t) &\overset{\mathcal{L}}{\longleftrightarrow} 1,\\ 1(t) &\overset{\mathcal{L}}{\longleftrightarrow} 1/s,\\ t &\overset{\mathcal{L}}{\longleftrightarrow} 1/s^2,\\ \sin(\omega t) &\overset{\mathcal{L}}{\longleftrightarrow} \omega/(s^2+\omega^2),\\ e^{-at} &\overset{\mathcal{L}}{\longleftrightarrow} 1/(s+a).\end{aligned}$$

Remark: $1(t)$ is the unit step function.

2.8 Transfer functions

Let $h(t)$ be the impulse response of a linear system.

$$Y(s) = \mathcal{L}\{y(t)\}, \qquad U(s) = \mathcal{L}\{u(t)\}, \qquad H(s) = \mathcal{L}\{h(t)\}.$$

Then

$$Y(s) = H(s)U(s).$$

2.9 State Space

LTI

$$\begin{aligned}\dot{x} &= Ax + Bu, \quad x(0) = x_0,\\ y &= Cx + Du.\end{aligned}$$

Transfer function:

$$H(s) = C(sI - A)^{-1}B + D.$$

System response:

$$Y(s) = \underbrace{H(s)U(s)}_{\text{Input response}} + \underbrace{C(sI-A)^{-1}x(0)}_{\text{Initial conditions response}}.$$

MIMO comes for free!