Frequency Response

We now know how to analyze and design ccts via s-domain methods which yield dynamical information

- Zero-state response
- Zero-input response
- Natural response
- Forced response

The responses are described by the exponential modes

The modes are determined by the poles of the response Laplace Transform

We next will look at describing cct performance via frequency response methods

This guides us in specifying the cct pole and zero positions

Sinusoidal Steady-State Response

Consider a stable transfer function with a sinusoidal input $v(t)=A\cos(\omega t)$ $V(s)=\dfrac{A\omega}{s^2+\omega^2}$

The Laplace Transform of the response has poles

- Where the natural cct modes lie
  - These are in the open left half plane Re(s)<0
- At the input modes $s=+j\omega$ and $s=-j\omega$

Only the response due to the poles on the imaginary axis remains after a sufficiently long time

This is the sinusoidal steady-state response

Sinusoidal Steady-State Response contd

Input $x(t) = A\cos(\omega t + \phi) = A\cos\omega t\sin\phi - A\sin\omega t\cos\phi$

Transform $X(s) = A\cos\phi \dfrac{s}{s^2+\omega^2} + A\sin\phi \dfrac{\omega}{s^2+\omega^2}$

Response Transform

$$Y(s) = T(s)X(s) = \frac{k}{s-j\omega} + \frac{k^*}{s+j\omega} + \frac{k_1}{s-p_1} + \frac{k_2}{s-p_2} + \cdots + \frac{k_N}{s-p_N}$$

Response Signal

$$y(t) = \underbrace{ke^{j\omega t} + k^* e^{-j\omega t}}_{\text{forced response}} + \underbrace{k_1 e^{p_1 t} + k_2 e^{p_2 t} + \cdots + k_N e^{p_N t}}_{\text{natural response}}$$

Sinusoidal Steady State Response $y_{SS}(t) = ke^{j\omega t} + k^* e^{-j\omega t}$

Sinusoidal Steady-State Response contd

Calculating the SSS response to $x(t) = A\cos(\omega t + \phi)$

Residue calculation

$$k = \lim_{s \to j\omega} \left[(s - j\omega)Y(s)\right] = \lim_{s \to j\omega} \left[(s - j\omega)T(s)X(s)\right]$$

$$= \lim_{s \to j\omega} \left[T(s)(s - j\omega)A\frac{s\cos\phi - \omega\sin\phi}{(s - j\omega)(s + j\omega)}\right] = T(j\omega)A\left[\frac{j\omega\cos\phi - \omega\sin\phi}{2j\omega}\right]$$

$$= \frac{1}{2}Ae^{j\phi}T(j\omega) = \frac{1}{2}A\left|T(j\omega)\right|e^{j(\phi + \angle T(j\omega))}$$

Signal calculation

$$y_{SS}(t) = \mathcal{L}^{-1}\left\{\frac{k}{s - j\omega} + \frac{k^*}{s + j\omega}\right\}$$

$$= |k|e^{j\angle k}e^{j\omega t} + |k|e^{-j\angle k}e^{-j\omega t} = 2|k|\cos(\omega t + \angle k)$$

$$y_{SS}(t) = A\left|T(j\omega)\right|\cos\left(\omega t + \phi + \angle T(j\omega)\right)$$

Sinusoidal Steady-State Response contd

Response to $x(t) = A\cos(\omega t + \phi)$

is $y_{SS}(t) = A|T(j\omega)|\cos\left(\omega t + \phi + \angle T(j\omega)\right)$

Output frequency = input frequency

Output amplitude = input amplitude $\times$ $|T(j\omega)|$

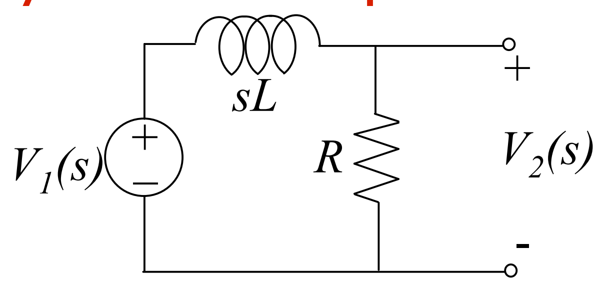
Output phase = input phase + $\angle T(j\omega)$

The Frequency Response of the transfer function T(s) is given by its evaluation as a function of a complex variable at $s=j\omega$

- We speak of the amplitude response and of the phase response
  - They cannot independently be varied
    - Bode's relations of analytic function theory

Example 11-13 T&R p 527

Find the steady state output for $v_1(t)=A\cos(\omega t+\phi)$



Compute the s-domain transfer function T(s)

Voltage divider $T(s) = \dfrac{R}{sL+R}$

Compute the frequency response

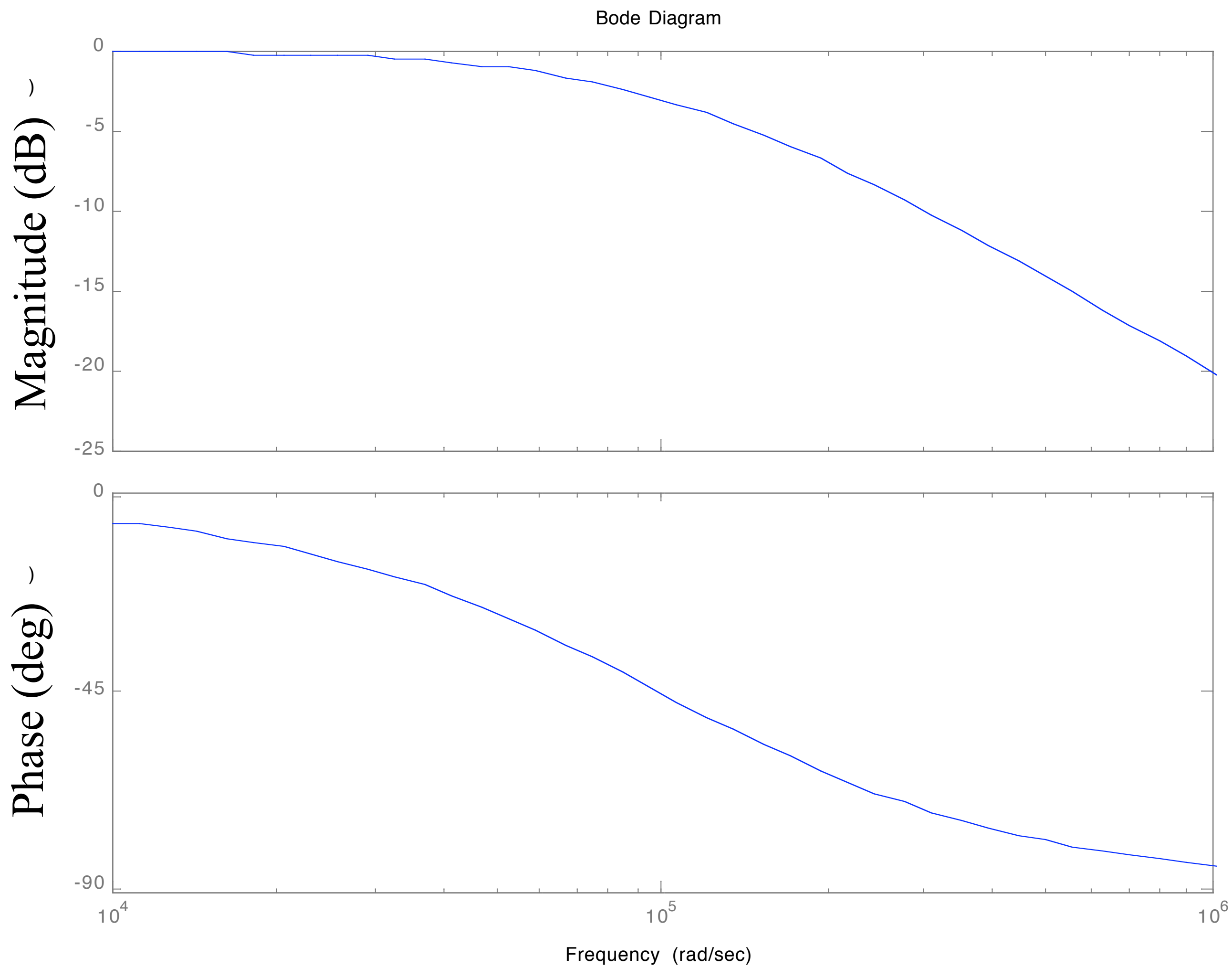
$$|T(j\omega)| = \frac{R}{\sqrt{R^2 + (\omega L)^2}}, \qquad \angle T(j\omega) = -\tan^{-1}\left(\frac{\omega L}{R}\right)$$

Compute the steady state output

$$v_{2SS}(t) = \frac{AR}{\sqrt{R^2 + (\omega L)^2}} \cos\left[\omega t + \phi - \tan^{-1}(\omega L / R)\right]$$

Frequency Responses – Bode Diagrams

Log-log plot of mag(T), log-linear plot arg(T) versus ω



Matlab Commands for Bode Diagram

Specify component values

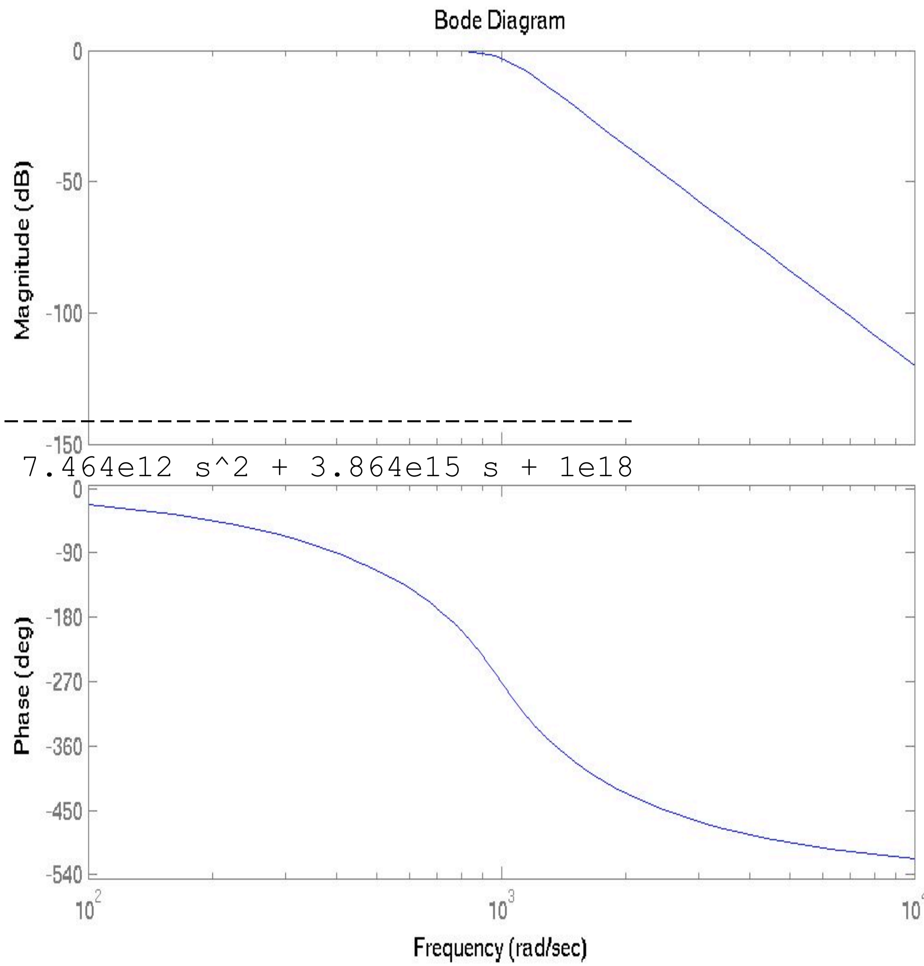
Set up transfer function

```
>> R=1000;L=0.01;
>> Z=tf(R,[L R])

Transfer function:
    1000
-------------
0.01 s + 1000

>> bode(Z)
```

Frequency Response Descriptors

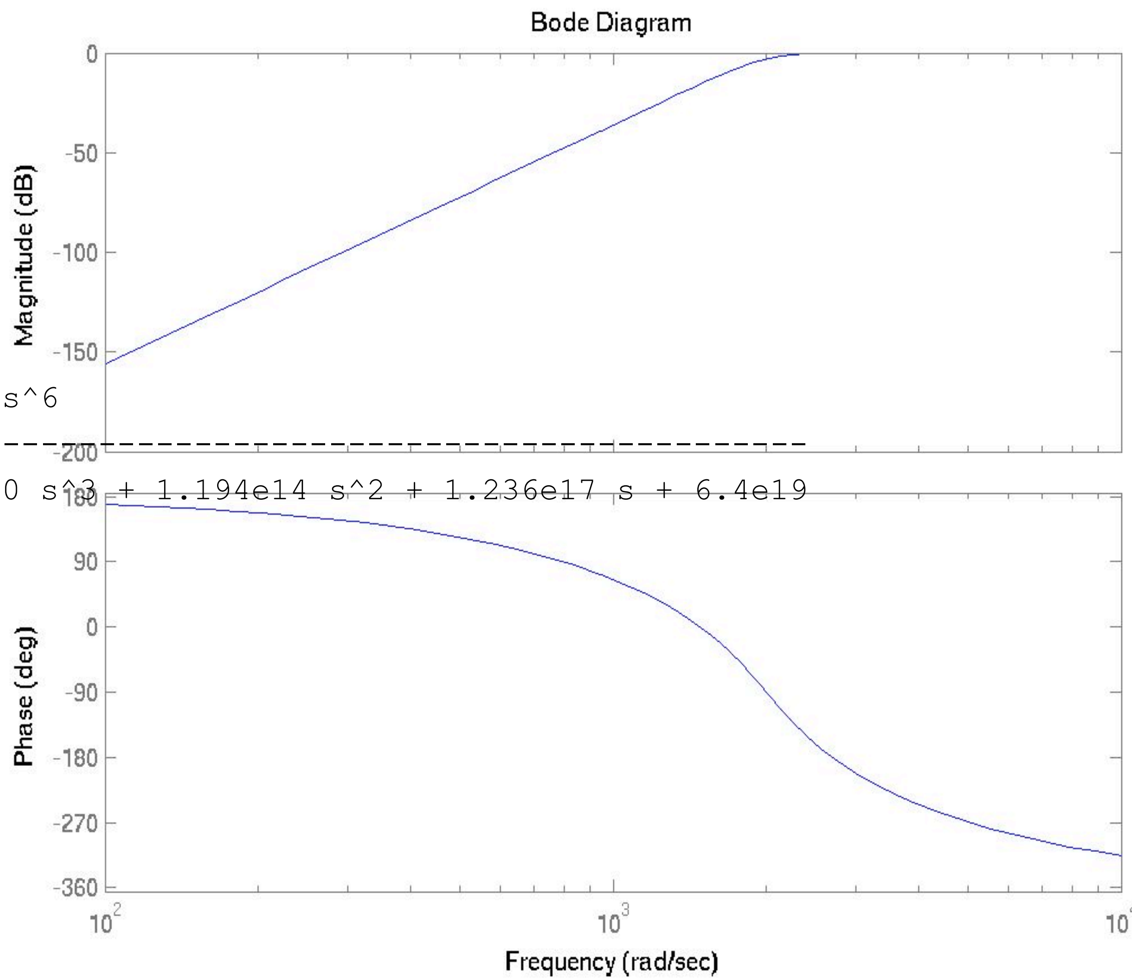
Lowpass Filters

```
[num,den]=butter(6,1000,'s');
lpass=tf(num,den);
lpass

Transfer function:
                                     1e18
-------------------------------------------------------------------------------
s^6 + 3864 s^5 + 7.464e06 s^4 + 9.142e09 s^3 + 7.464e12 s^2 + 3.864e15 s + 1e18

bode(lpass)
```

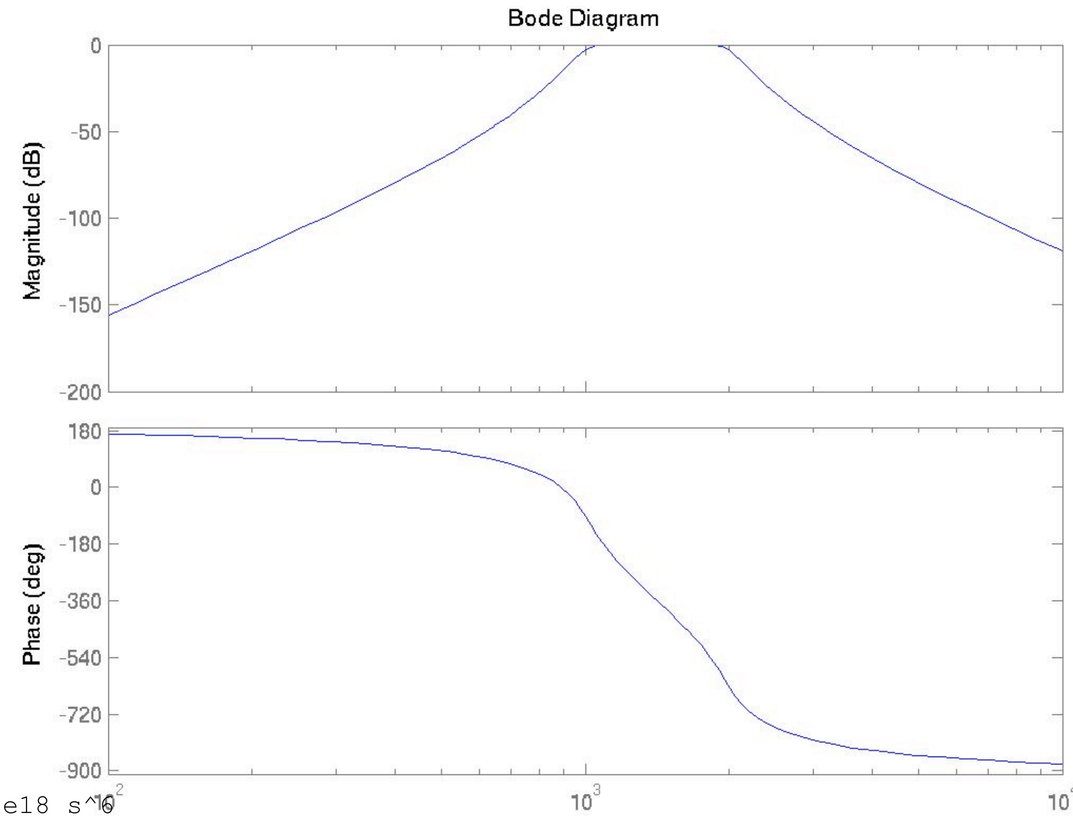


High Pass Filters

```
[num,den]=butter(6,2000,'high','s');
hpass=tf(num,den)

Transfer function:
                                                s^6
------------------------------------------------------------------------------------------
s^6 + 7727 s^5 + 2.986e07 s^4 + 7.313e10 s^3 + 1.194e14 s^2 + 1.236e17 s + 6.4e19

bode(hpass)
```

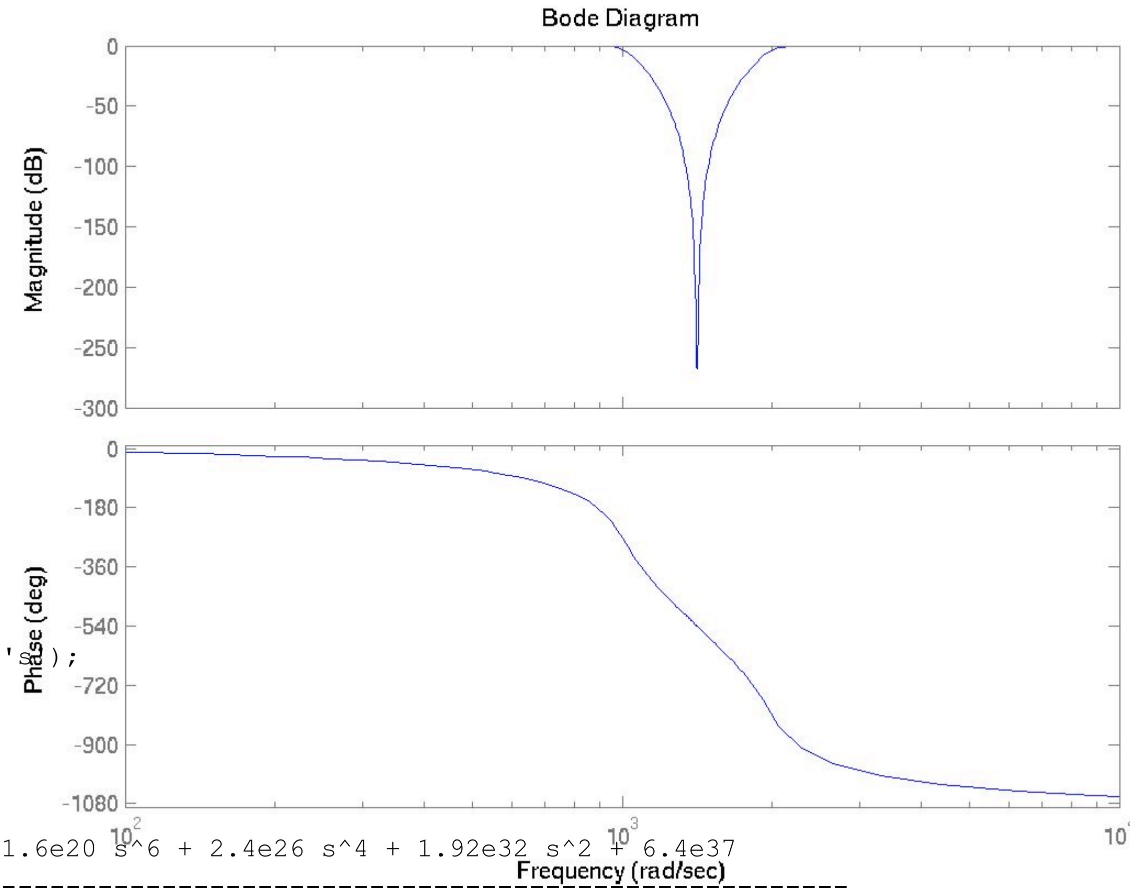


Bandpass Filters




```
[num,den]=butter(6,[1000 2000],'s');
bpass=tf(num,den)

Transfer function:

                                                1e18 s^6
------------------------------------------------------------------------------------------
s^12 + 3864 s^11 + 1.946e07 s^10 + 4.778e10 s^9 + 1.272e14 s^8 + 2.133e17 s^7 + 3.7e20 s^6

        + 4.265e23 s^5 + 5.087e26 s^4 + 3.822e29 s^3 + 3.114e32 s^2 + 1.236e35 s + 6.4e37


bode(bpass)
```

Bandstop Filters




```
[num,den]=butter(6,[1000 2000],'stop','s');
bstop=tf(num,den)

Transfer function:

      s^12 + 1.2e07 s^10 + 6e13 s^8 + 1.6e20 s^6 + 2.4e26 s^4 + 1.92e32 s^2 + 6.4e37
---------------------------------------------------------------------------------------------
s^12 + 3864 s^11 + 1.946e07 s^10 + 4.778e10 s^9 + 1.272e14 s^8 + 2.133e17 s^7 + 3.7e20 s^6

        + 4.265e23 s^5 + 5.087e26 s^4 + 3.822e29 s^3 + 3.114e32 s^2 + 1.236e35 s + 6.4e37


bode(bstop)
```