MAE 140 – Linear Circuits – Winter 2008
Final

Instructions

1) This exam is open book. You may use whatever written materials you choose, including your class notes and textbook. You may use a hand calculator with no communication capabilities.

2) You have 170 minutes.

3) On the questions for which we have given the answers, please provide detailed derivations.

Question 1 — Equivalent Circuits

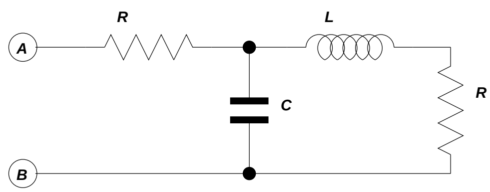

Figure 1: Circuit for Question 1 (i)

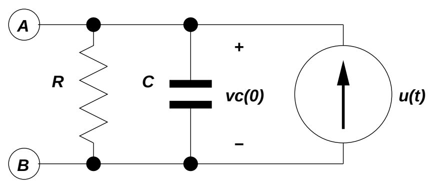

Figure 2: Circuit for Question 1 (ii)

Part (i) [5 marks] Assuming zero initial conditions, find the impedance equivalent to the circuit in Figure 1 as seen from terminals A and B. The answer should be given as a ratio of two polynomials.

Compute

$$
\begin{aligned}
Z(s) &= R + [1/(sC) // (R + sL)] \\
&= R + \frac{(R + sL)/(sC)}{R + sL + 1/(sC)} \\
&= R + \frac{R + sL}{s^2LC + sRC + 1} \\
&= \frac{R(s^2LC + sRC + 1) + R + sL}{s^2LC + sRC + 1} \\
&= \frac{s^2RLC + s(R^2C + L) + 2R}{s^2LC + sRC + 1}
\end{aligned}
$$

[+5 marks; partial marks for partial work]

Part (ii) [5 marks] Assuming that the initial condition of the capacitor is as indicated in the diagram, redraw the circuit shown in Figure 2 in s-domain. Then use source transformations to find the s-domain Norton equivalent to the circuit as seen from terminals A and B.

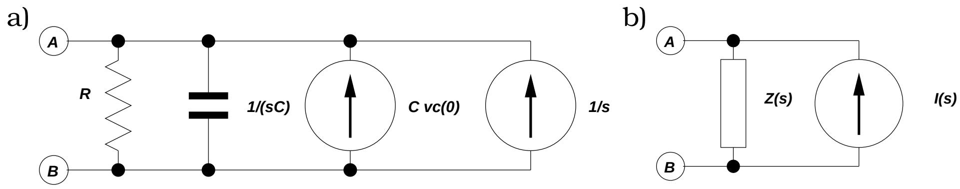

Figure 3: Sequence of Transformations – Question 1 (ii)

First transform the circuit to the s-domain to include the current source $Cv_C(0)$ in parallel with $1/(sC)$ as in Figure 3a).

[+2 marks]

Associate the capacitor and resistor into the impedance

$$Z(s) = \frac{\frac{R}{sC}}{R + \frac{1}{sC}} = \frac{R}{1 + sRC}$$

and add the two current sources into a single current source

$$I(s) = 1/s + Cv_C(0)$$

The Norton equivalent circuit is then shown in Figure 3b).

[+3 marks]

Question 2 — Laplace Domain Circuit Analysis

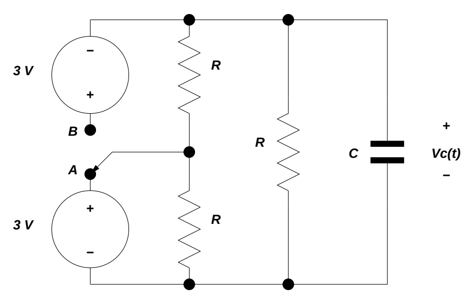

Figure 4: RC circuit for Laplace analysis.

Part (i) [2 marks] Consider the circuit depicted in Figure 4. The voltage sources are constant. The switch is kept in position A for a very long time. At $t = 0$ it is moved to position B. Show that the initial capacitor voltage is given by

$$v_C(0^-) = 1.5V.$$

[Show your working.]

With a constant input and the switch in position A for a very long time, the circuit reaches steady state. In steady state, the circuit is described by all time derivatives of signals being zero. In particular, $\frac{dv_C(t)}{dt} = 0$. Therefore, the steady state current through the capacitor is zero, as is the current through the resistor R_1. In this configuration the resistor R in parallel with C and the resistor R in parallel with the upper voltage source act as a voltage divider yielding $v_C(0^-) = 3R/(R + R) = 1.5$ V.

[+2 marks]

Part (ii) [2 marks] Use this initial condition to transform the circuit into the s-domain for $t \geq 0$. Use an equivalent model for the capacitor in which the initial condition appears in a current source.
[Show your working.]

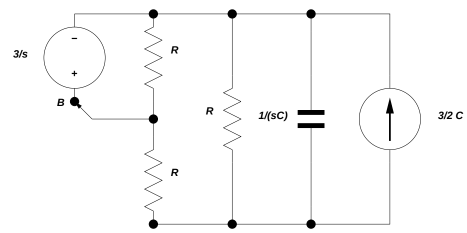

Figure 5: s-domain circuit.

With the initial voltage from Part (i), redraw the circuit for time $t \geq 0$ as in Figure 5. Note that the initial capacitor voltage $V_C(0)$ appears as a current source $3/2C$ in parallel with the impedance $1/(sC)$.

[+2 marks]

Part (iii) [3 marks] Use s-domain circuit analysis and inverse Laplace transforms to show that the capacitor voltage satisfies,

$$v_C(t) = 3\left(e^{-2t/(RC)} - 1/2\right) u(t)$$

Hint: simplify the circuit, transform all voltage sources into equivalent current sources and use nodal-analysis.

[Show your working.]

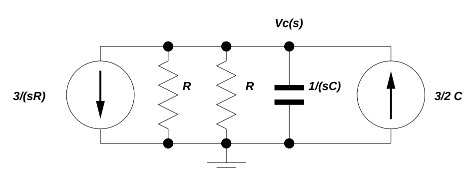

Figure 6: Simplified s-domain circuit.

First eliminate the resistor in parallel with the voltage source from the circuit as its current will not affect the voltage in the capacitor, then transform the voltage source which is now in series with a resistor R obtaining the circuit in Figure 6.

[+1 mark]

Choose a reference node and write node-voltage equations by inspection

$$\left(\frac{2}{R} + sC\right) V_C(s) = \frac{3}{2}C - \frac{3}{sR}.$$

Dividing by C

$$(s + 2/(RC))\, V_C(s) = \frac{3}{2s}\,(s - 2/(RC))$$

then

$$V_C(s) = \frac{3}{2s}\,\frac{s - 2/(RC)}{s + 2/(RC)}.$$

[+1 mark]

By partial fraction expansion

$$V_C(s) = \frac{\alpha_1}{s} + \frac{\alpha_2}{s + 2/(RC)}$$

where

$$\alpha_1 = \lim_{s\to 0} \frac{3}{2}\,\frac{s - 2/(RC)}{s + 2/(RC)} = -\frac{3}{2}, \qquad \alpha_2 = \lim_{s\to -2/(RC)} \frac{3(s - 2/(RC))}{2s} = 3$$

that is

$$V_C(s) = \frac{3}{s + 2/(RC)} - \frac{3}{2s}$$

Hence, taking inverse Laplace transforms,

$$v_C(t) = \left(3e^{-2t/(RC)} - 3/2\right) u(t).$$

[+1 mark]

Part (iv) [3 marks] Use the final value theorem to show that the voltage on the capacitor after the switch is kept in position B for a very long time is given by

$$v_C(\infty) = -1.5V.$$

Verify whether the same solution is obtained from the answer to the previous question.

[Show your working.]

From the answer to Part (iii)

$$V_C(s) = \frac{3}{2s}\,\frac{s - 2/(RC)}{s + 2/(RC)}.$$

Then

$$\lim_{t\to\infty} v_C(\infty) = \lim_{s\to 0} sV_C(s) = \lim_{s\to 0} \frac{3}{2}\,\frac{s - 2/(RC)}{s + 2/(RC)} = -\frac{3}{2}V.$$

[+2 marks]

Also from Part (iii)

$$v_C(t) = \left(3e^{-2t/(RC)} - 3/2\right) u(t).$$

so that

$$\lim_{t\to\infty} v_C(\infty) = \lim_{t\to\infty} \left(3e^{-2t/(RC)} - 3/2\right) u(t) - \frac{3}{2}V$$

which gives the same answer.

[+1 mark]

Question 3 — Active Filter Analysis and Design

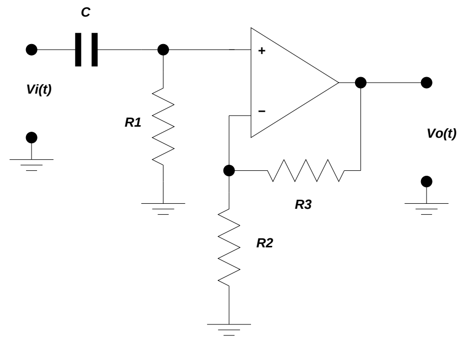

Figure 7: Filter for analysis

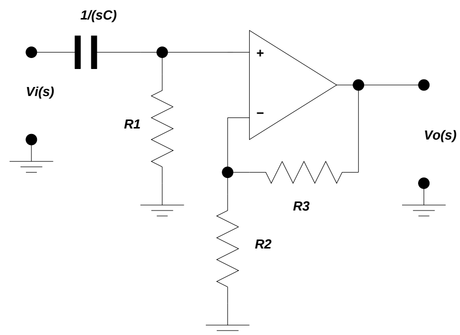

Figure 8: Filter in the s-domain

Part (i) [4 marks] Assuming zero initial conditions, transform the circuit in the Figure 7 into the s-domain and compute the transfer function from $V_i(s)$ to $V_o(s)$.

First transform the circuit into the s-domain as in Figure 8.

[+1 mark]

Then

$$V_p(s) = \frac{R_1}{R_1 + 1/(sC)} V_i(s)$$

and

$$V_o(s) = \frac{R_2 + R_3}{R_2} V_p(s)$$

[+2 marks]

Therefore

$$T(s) = V_i(s)/V_o(s) = \frac{R_2 + R_3}{R_2} \frac{s}{s + 1/(R_1 C)}$$

[+1 mark]

Part (ii) [3 marks] Showing your reasoning, determine the nature of this filter's frequency response. Further, determine the gain of the filter and its cut-off frequency.

This is a first order filter so that the filter's characteristic can be obtained evaluating $T(j\omega)$ as $\omega \to 0$ and $\omega \to \infty$. In this case

$$\lim_{\omega \to 0} T(j\omega) = 0, \qquad \lim_{\omega \to \infty} T(j\omega) = \frac{R_2 + R_3}{R_2}.$$

Therefore it is a high-pass *filter.*

[+1 mark]

The cut-off frequency coincides with the absolute values of the pole

$$\omega_c = 1/(R_1 C).$$

[+1 mark]

The gain is the peak value of the absolute value of the frequency response which is at $\omega \to \infty$

$$K = \lim_{\omega\to\infty} T(j\omega) = \frac{R_2 + R_3}{R_2}.$$

[+1 mark]
[You also get marks if you associate the transfer function with the expressions in the book and invoque the book's results, as long as they are correct.]

Part (iii) [3 marks] If $C = 100$nF, find values of R_1, R_2 and R_3 so that the cutoff frequency is 10 KHz and the filter gain is 2.

From the answer to Part (ii)

$$\omega_c = \frac{1}{R_1 C} = 2\pi 10^4 \quad \Longrightarrow \quad R_1 = \frac{1}{2\pi 10^4 \times 100 \times 10^{-9}} = \frac{1}{2\pi}\mathrm{K}\,\Omega$$

[+1 mark]

Also

$$K = \frac{R_2 + R_3}{R_2} = 2 \Longrightarrow R_2 = R_3.$$

[+1 mark]

For instance, $R_2 = R_3 = 10$ *K*Ω*.*

[+1 mark]

Question 4 — Laplace Domain Circuit Analysis

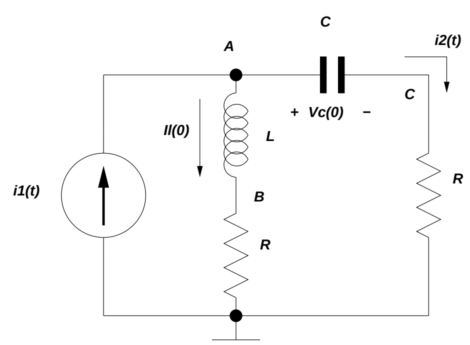

Figure 9: Frequency Response Circuit

[6 marks] Formulate node-voltage equations for the circuit in Figure 9. Use the reference node and other labels as shown in the figure. Do not assume zero initial conditions!

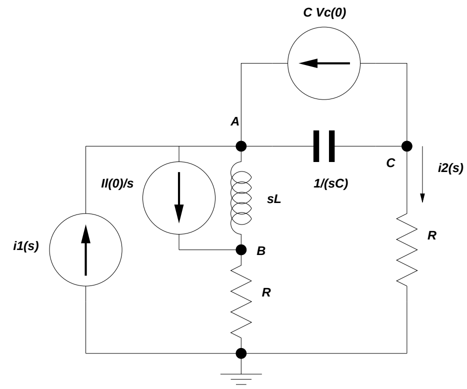

Figure 10: Circuit in the s-domain

First transform the circuit into the s-domain as in Figure 10.

[+2 marks for correct circuit]
[+1 mark for correct initial conditions]

Then write the node-voltage equations by inspection

$$\begin{bmatrix} \frac{1}{sL}+sC & -\frac{1}{sL} & -sC \\ -\frac{1}{sL} & \frac{1}{sL}+\frac{1}{R} & 0 \\ -sC & 0 & sC+\frac{1}{R} \end{bmatrix} \begin{pmatrix} V_A(s) \\ V_B(s) \\ V_C(s) \end{pmatrix} = \begin{pmatrix} I_1(s)+Cv_C(0)-i_L(0)/s \\ i_L(0)/s \\ -Cv_C(0) \end{pmatrix}$$

or one-by-one using KCL.

[+3 marks for correct equations]

Question 5 — Frequency Response

[4 marks] Let $L = (1/\pi)$ H, $C = (25/\pi)$ μF and $R = 200$ Ω. Using what you known about frequency response, find the steady state current $i_2(t)$ in the circuit given in Figure 9 when the input current is $i_1(t)$ is a cosine function with amplitude '1' A and frequency 100 Hz.

First compute the transfer function from $I_1(s)$ to $I_2(s)$ using current division:

$$I_2(s) = \frac{\dfrac{1}{R+1/(sC)}}{\dfrac{1}{R+1/(sC)}+\dfrac{1}{R+sL}} I_1(s) = \frac{R+sL}{2R+1/(sC)+sL} I_1(s) = \frac{sC(R+sL)}{1+s2RC+s^2LC} I_1(s).$$

Then

$$T(s) = \frac{I_2(s)}{I_1(s)} = \frac{sC(R+sL)}{1+s2RC+s^2LC}.$$

[+2 marks]

Now evaluate $T(s)$ at $s = j\omega$

$$T(j\omega) = \frac{j\omega C(R+j\omega L)}{(1-\omega^2 LC)+j\omega 2RC}$$

at $\omega = 200\pi$ for $L = (1/\pi)$ H, $C = (25/\pi) \times 10^{-6}$ F and $R = 200\ \Omega$. Some auxiliary computations

$$\begin{aligned} R + j\omega L &= 200 + j\pi 200/\pi = 200(1+j) \\ 1 - \omega^2 LC &= 1 - 4\times 10^4 \times \pi^2 \times 25 \times 10^{-6}/\pi^2 = 0 \\ j\omega 2RC &= j400 \times \pi \times 200 \times 25 \times 10^{-6}/\pi = 2j \\ j\omega C &= j200 \times \pi \times 25 \times 10^{-6}/\pi = j5 \times 10^{-3} \end{aligned}$$

so that

$$T(j200\pi) = \frac{j5\times 10^{-3} \times 200(1+j)}{2j} = \frac{1}{2}(1+j) = \frac{\sqrt{2}}{2}\angle \pi/4.$$

[+1 mark for evaluating $T(j200\pi)$]

Now evaluate response in steady state

$$i_2^{ss}(t) = |T(j200\pi)|\cos(200\pi t + \angle T(j200\pi)) = \frac{\sqrt{2}}{2}\cos(200\pi t + \pi/4)\ \text{A}.$$

[+1 mark for $i_2^{ss}(t)$]
[Other approaches are possible and will be taken into consideration]

Question 6 — Op-Amp Analysis and Application (Bonus)

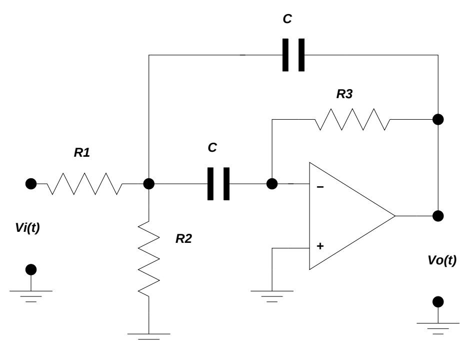

Figure 11: Op-Amp Circuit

Part (i) [4 marks] Using the fundamental op-amp relationships, find the transfer function from $V_i(s)$ to $V_o(s)$ in Figure 11. Assume zero initial conditions.

Write KCL at node '–'

$$sC\left(V_n(s) - V_A(s)\right) + \frac{1}{R_3}\left(V_n(s) - V_o(s)\right) = 0$$

Now use the fact that $V_n(s) = V_p(s) = 0$ to write

$$V_o(s) = -(sR_3C)V_A(s).$$

(You can also arrive at this using the inverter op-amp relationship directly!)

[+1 mark]

Now write KCL at node A

$$\frac{1}{R_1}\left(V_A(s)-V_i(s)\right)+\frac{1}{R_2}V_A(s)+sCV_A(s)+sC\left(V_A(s)-V_o(s)\right)=0$$

with $V_n(s)=0$.

[+1 mark]

Put the two together to write

$$\left(\frac{1}{R_1}+\frac{1}{R_2}+2sC\right)V_A(s)-sCV_o(s)=\frac{1}{R_1}V_i(s)$$

so that

$$\left(\frac{1}{R_1}+\frac{1}{R_2}+2sC+s^2C^2R_3\right)V_A(s)=\frac{1}{R_1}V_i(s)$$

[+1 mark]

Now substitute $V_A(s)$ for $V_i(s)$

$$\begin{aligned}V_o(s)&=-\frac{sR_3C}{R_1}\left(\frac{1}{R_1}+\frac{1}{R_2}+2sC+s^2C^2R_3\right)^{-1}V_i(s)\\&=-\frac{s}{R_1C}\left(\frac{R_1+R_2}{C^2R_3R_1R_2}+2\frac{1}{CR_3}s+s^2\right)^{-1}V_i(s)\end{aligned}$$

To help with the algebra define

$$\omega_0^2=\frac{R_1+R_2}{C^2R_3R_1R_2}\qquad\qquad\beta=\frac{1}{CR_3}$$

so that

$$V_o(s)=-\frac{s}{R_1C(\omega_0^2+2\beta s+s^2)}V_i(s)$$

[+1 mark]

Part (ii) [3 marks] Calculate the input and the output impedances of this circuit.

The output impedance is the output impedance of the opAmp, which is close to zero.

[+ 1 mark]

The current seen from the source $V_i(s)$ is the current on the resistor R_1

$$I_i(s)=\frac{1}{R_1}(V_i(s)-V_A(s))$$

and the input impedance is

$$Z(s)=\frac{V_i(s)}{I_i(s)}=\frac{R_1V_i(s)}{V_i(s)-V_a(s)}=\frac{R_1}{1-V_A(s)/V_i(s)}$$

[+ 2 marks]

From the answer to Part (i)

$$\frac{V_A(s)}{V_i(s)}=\frac{1}{C^2R_1R_3(\omega_0^2+2s\beta+s^2)}$$

and

$$1 - \frac{V_A(s)}{V_i(s)} = \frac{[C^2R_1R_3\omega_0^2 - 1] + 2C^2R_1R_3\beta s + C^2R_1R_3s^2}{C^2R_1R_3(\omega_0^2 + 2\beta s + s^2)} = \frac{1 - 1/(C^2R_1R_3\omega_0^2) + 2\beta s + s^2}{\omega_0^2 + 2\beta s + s^2}$$

so that

$$Z(s) = \frac{R_1(\omega_0^2 + 2\beta s + s^2)}{1 - 1/(C^2R_1R_3\omega_0^2) + 2\beta s + s^2}.$$

Part (iii) [3 marks] Set $R_1 = 1100\ \Omega$, $R_2 = 1200\ \Omega$, $R_3 = 150$ K Ω, $C = .01\ \mu$F. Tell me as much as you can about this circuit's function.

First compute ω_0^2 and β

$$\omega_0^2 = \frac{R_1 + R_2}{C^2R_3R_1R_2} = \frac{2.3 \times 10^3}{10^{-16} \times 1.5 \times 10^5 \times 1.1 \times 1.2 \times 10^6} = \frac{2.3 \times 10^3}{1.98 \times 10^{-5}} = 1.16 \times 10^8$$

$$\beta = \frac{1}{CR_3} = \frac{1}{1.5 \times 10^5 \times 10^{-8}} = \frac{2}{3}10^3 = 0.67 \times 10^3$$

Now let's look at the poles of

$$\omega_0^2 + 2\beta s + s^2 = 0$$

which are of the form

$$\begin{aligned} s &= -\beta \pm \sqrt{\beta^2 - \omega_0^2} = -0.67 \times 10^3 \pm \sqrt{0.44 \times 10^6 - 1.16 \times 10^8} \\ &= -0.67 \times 10^3 \pm \sqrt{-1.15 \times 10^8} \\ &= -0.67 \times 10^3 \pm j1.07 \times 10^4 \end{aligned}$$

So this operates like an RLC circuit, with two complex conjugate poles at

$$-0.67 \times 10^3 \pm j1.07 \times 10^4.$$

It could be a band-pass or a band-stop filter.

[+ 1 mark]

For the qualitative frequency response let us look at $s \to 0$ and $s \to \infty$. From Part (i)

$$T(s) = \frac{V_o(s)}{V_i(s)} = -\frac{s}{R_1C(\omega_0^2 + 2\beta s + s^2)}$$

so that

$$\lim_{s\to 0} T(s) = 0, \qquad \lim_{s\to\infty} T(s) = 0$$

so that it must be a band-pass *filter.*

[+ 1 mark]

For other quantitative information we can use directly the expressions derived in the book. For example, the center frequency is

$$\omega_c = \omega_0 = \sqrt{1.16 \times 10^8} = 1.07 \times 10^4 = 10700 \text{ rad/s} = 1702 \text{ Hz}.$$

The bandwith of the filter is

$$B = 2\beta = 3.34 \times 10^3 = 3340 \text{ rad/s} = 531 \text{ Hz}.$$

The Q of the filter is

$$Q = \frac{\omega_0}{B} = \frac{1702}{531} = 3.2$$

[+ 1 mark]