Question 1 [Circuit variables]
You went to the store and bought a 100W bulb. When you got home you hooked it in the socket and noticed that the light was very dim. You then checked the bulb and discovered that the bulb was rated 100W/220V.

a) [2 marks] What is the resistance and the power dissipated by the wrong bulb at your home? (Consider the voltage at your home to be 127V)

The rated power should give you the resistance \( R \) since

\[
p = vi = v(v/R) = v^2/R \implies R = v^2/p = 220^2/100 \Omega = 484 \Omega
\]

When I plugged it in at home I got an actual power dissipation of

\[
p = v^2/R = 127^2/484 W \approx 33 W
\]

b) [1 mark] What is the resistance of a bulb rated 100W/127V?

As before, the rated power should give you the resistance

\[
R = v^2/p = 127^2/100 \Omega \approx 161 \Omega
\]
**Question 2 [Voltage divider]**

Regarding the following voltage-divider circuit

![Voltage Divider Circuit Diagram]

**a)** [2 marks] Show that if \( R_1 = R_3 = R \Omega \) and \( R_2 = 2R \Omega \) then \( v_A - v_B = v_C - v_D = 2.5 \text{V} \) and \( v_B - v_C = 5 \text{V} \).

\[ i = \frac{10}{(R_1 + R_2 + R_3)} = \frac{10}{4R} = (2.5/R) \text{A} \]

*The current flowing through all the resistors is*

*Therefore the voltages across the resistors can be computed*

\[ v_A - v_B = R_1 i = R(2.5/R) = 2.5 \text{V} \]
\[ v_B - v_C = R_2 i = 2R(2.5/R) = 5.0 \text{V} \]
\[ v_C - v_D = R_3 i = R(2.5/R) = 2.5 \text{V} \]

**b)** [2 marks] Use Thevenin’s equivalence to show that at terminals \( B \) and \( C \) the equivalent resistance is \( R \Omega \). Draw the equivalent Thevenin circuit.

*First show that the open circuit voltage \( v_T = v_B - v_C = 5.0 \text{V} \) as computed in item a).*

*Then compute the short circuit current \( i_N \) which is the current flowing through resistors \( R_1 \) and \( R_3 \) when \( R_2 \) is short circuited*

\[ i_N = \frac{10}{(R_1 + R_3)} = \frac{10}{2R} = (5.0/R) \text{A} \]

*Therefore the equivalent resistance is*

\[ R_{EQ} = \frac{v_T}{i_N} = \frac{5.0/(5.0/R)}{R} = R \Omega \]
The equivalent Thevenin circuit is

\[ \text{The equivalent Norton circuit is} \]

\[ \frac{5}{R} \text{A} \]

\(R\) \(B\)

\(C\)

c) [1 mark] Draw the Norton equivalent circuit as seen from terminals \(B\) and \(C\).
Question 3 [Node-voltage analysis]
Regarding the following circuit

a) [2 marks] Transform the series connection of the voltage source $v_2$ and the resistor $R_5$ into an equivalent current source in parallel with a resistance and draw the resulting circuit.

*b) [3 marks] Formulate node-voltage equations for the resulting circuit. Remember to properly choose the ground node! Clearly state the unknowns and the equations to be solved.*

First note that there is still one source of voltage that cannot be transformed into a source of
current. In order to handle that choose the ground to be node A, as indicated in the figure above. C is an alternative.

Now write the node-voltage equations by inspection

\[
\begin{bmatrix}
\frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4} & -\frac{1}{R_3} & -\frac{1}{R_4} \\
-\frac{1}{R_3} & \frac{1}{R_3} + \frac{1}{R_5} & -\frac{1}{R_5} \\
-\frac{1}{R_4} & -\frac{1}{R_5} & \frac{1}{R_1} + \frac{1}{R_4} + \frac{1}{R_5}
\end{bmatrix}
\begin{bmatrix}
-v_B \\
v_C \\
v_D
\end{bmatrix}
= \begin{bmatrix}
0 \\
-i_{v_1} + \frac{v_2}{R_5} \\
-\frac{v_2}{R_5}
\end{bmatrix}
\]

where \(i_{v_1}\) is the current on the voltage source \(v_1\).

Having set A as ground, \(v_C = v_1\) is now known and can be eliminated from the vector of unknowns. Furthermore, the presence of the current \(i_{v_1}\) on the right hand side of the second equation means that this equation can be solved after \(v_B\) and \(v_D\) are determined. This results in the smaller set of equations

\[
\begin{bmatrix}
\frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4} & -\frac{1}{R_4} \\
-\frac{1}{R_4} & \frac{1}{R_1} + \frac{1}{R_4} + \frac{1}{R_5}
\end{bmatrix}
\begin{bmatrix}
v_B \\
v_D
\end{bmatrix}
= \begin{bmatrix}
\frac{v_1}{R_3} \\
\frac{v_1}{R_5} - \frac{v_2}{R_5}
\end{bmatrix}
\]

to be solved for \(v_B\) and \(v_D\).

c) [BONUS - 1 mark] Can you formulate node-voltage equations for the above circuit without transforming the voltage source \(v_2\) into an equivalent current source?

One way is using supernodes. We will do that later. Another way is noting that we can commute \(v_2\) with \(R_2\) to produce the following equivalent circuit
Now write the node-voltage equations by inspection. Because $v_A = -v_1$ and $v_E = -v_2$ are known we need only to write the current equations at nodes B and D. This results in

\[
\begin{bmatrix}
-\frac{1}{R_2} & \frac{1}{R_3} + \frac{1}{R_4} & -\frac{1}{R_4} & 0 \\
-\frac{1}{R_1} & \frac{1}{R_4} + \frac{1}{R_5} & -\frac{1}{R_5} & 0
\end{bmatrix}
\begin{bmatrix}
v_A \\
v_B \\
v_D \\
v_E
\end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

Having set C as ground, $v_A = -v_1$ and $v_E = -v_2$ are now known and can be eliminated from the vector of unknowns. This results in the smaller set of equations

\[
\begin{bmatrix}
\frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4} & -\frac{1}{R_4} & 0 \\
-\frac{1}{R_3} & \frac{1}{R_3} + \frac{1}{R_5} & -\frac{1}{R_5} \\
-\frac{1}{R_4} & 0 & \frac{1}{R_1} + \frac{1}{R_4} \\
0 & -\frac{1}{R_5} & 0
\end{bmatrix}
\begin{bmatrix}
v_B \\
v_C \\
v_D \\
v_E
\end{bmatrix}
= \begin{bmatrix} 0 \\ -i_{v_1} \\ -i_{v_2} \end{bmatrix}
\]

on the unknowns $v_B$ and $v_D$.

Now let us do it the alternative way: creating a supernode D-E on the original diagram. Setting the ground to node A we write node-voltage equations by inspection on the nodes B, C, D and E

\[
\begin{bmatrix}
\frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4} & -\frac{1}{R_3} & -\frac{1}{R_4} & 0 \\
-\frac{1}{R_3} & \frac{1}{R_3} + \frac{1}{R_5} & 0 & -\frac{1}{R_5} \\
-\frac{1}{R_4} & 0 & \frac{1}{R_1} + \frac{1}{R_4} & 0 \\
0 & -\frac{1}{R_5} & 0 & \frac{1}{R_5}
\end{bmatrix}
\begin{bmatrix}
v_B \\
v_C \\
v_D \\
v_E
\end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \\ -i_{v_1} \\ i_{v_2} \end{bmatrix}
\]

where $i_{v_1}$ is the current on the voltage source $v_1$ and $i_{v_2}$ is the current on the voltage source $v_2$.

The sum of the third and fourth equation provide the current balance for the supernode D-E. The second equation can be eliminated because of the presence of $i_{v_1}$. That reduces the problem to

\[
\begin{bmatrix}
\frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4} & -\frac{1}{R_3} & -\frac{1}{R_4} & 0 \\
-\frac{1}{R_3} & \frac{1}{R_3} + \frac{1}{R_5} & 0 & -\frac{1}{R_5} \\
-\frac{1}{R_4} & 0 & \frac{1}{R_1} + \frac{1}{R_4} & 0 \\
0 & -\frac{1}{R_5} & 0 & \frac{1}{R_5}
\end{bmatrix}
\begin{bmatrix}
v_B \\
v_C \\
v_D \\
v_E
\end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \\ -i_{v_1} \\ i_{v_2} \end{bmatrix}
\]

Having set A as ground, $v_C = v_1$ is now known and can be eliminated from the vector of unknowns. We can also eliminate $v_E$ by noticing that $v_E = v_D + v_2$.

\[
\begin{bmatrix}
\frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4} & -\frac{1}{R_4} \\
-\frac{1}{R_4} & \frac{1}{R_1} + \frac{1}{R_4} + \frac{1}{R_5}
\end{bmatrix}
\begin{bmatrix}
v_B \\
v_D
\end{bmatrix}
= \begin{bmatrix} \frac{v_1}{R_3} \\ \frac{v_1 - v_2}{R_5} \end{bmatrix}
\]
As a final remark, note that, at first, the two answers do not seem to be the same, as the right hand sides are different. However, recalling that C is the ground in the first approach, we see that \( v_B \) and \( v_D \) cannot be the same. Indeed, in the first approach what we have implicitly computed is

\[
\begin{bmatrix}
\frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4} & -\frac{1}{R_4} \\
-\frac{1}{R_4} & \frac{1}{R_1} + \frac{1}{R_4} + \frac{1}{R_5}
\end{bmatrix}
\begin{pmatrix}
(v_B - v_C) \\
v_D - v_C
\end{pmatrix}
= 
\begin{pmatrix}
-\frac{v_1}{R_2} \\
\frac{1}{R_1 - \frac{R_5}{R_5}}
\end{pmatrix}
\]

Now using the fact that \( v_C = v_A + v_1 \) so that

\[
\begin{bmatrix}
\frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4} & -\frac{1}{R_4} \\
-\frac{1}{R_4} & \frac{1}{R_1} + \frac{1}{R_4} + \frac{1}{R_5}
\end{bmatrix}
\begin{pmatrix}
(v_B - v_A) \\
v_D - v_A
\end{pmatrix}
= 
\begin{pmatrix}
-\frac{v_1}{R_2} \\
\frac{1}{R_1 - \frac{R_5}{R_5}}
\end{pmatrix}
+ 
\begin{pmatrix}
\frac{1}{R_2} + \frac{1}{R_3} \\
\frac{1}{R_1} + \frac{1}{R_5}
\end{pmatrix}
\begin{pmatrix}
\frac{v_1}{R_3} \\
\frac{v_1}{R_5} - \frac{v_2}{R_5}
\end{pmatrix}
\]

which proves that the two answers are indeed the same, if measured relative to the same reference.