MAE 140 – Linear Circuits – Summer 2007
Midterm Solution

Question 1 [Circuit variables]
You went to the store and bought a $100W$ bulb. When you got home you hooked it in the socket and noticed that the light was very dim. You then checked the bulb and discovered that the bulb was rated $100W/220V$.
a) [2 marks] What is the resistance and the power dissipated by the wrong bulb at your home? (Consider the voltage at your home to be $127V$)

The rated power should give you the resistance R since

$$p = vi = v(v/R) = v^2/R \quad \Rightarrow \quad R = v^2/p = 220^2/100\,\Omega = 484\,\Omega$$

When I plugged it in at home I got an actual power dissipation of

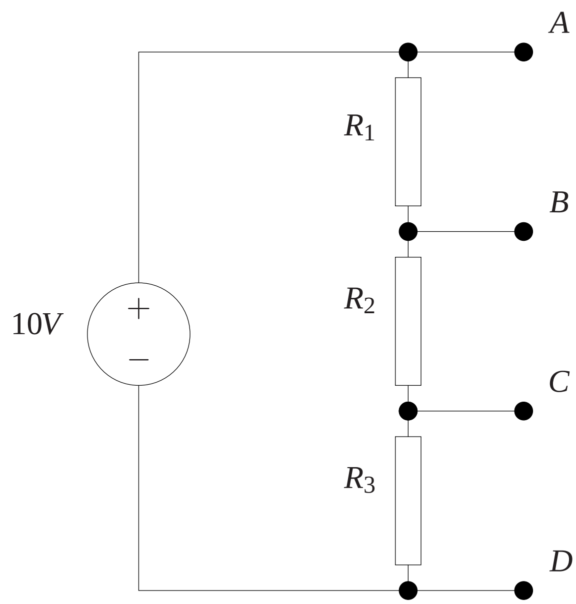
$$p = v^2/R = 127^2/484\,W \approx 33\,W$$

b) [1 mark] What is the resistance of a bulb rated $100W/127V$?

As before, the rated power should give you the resistance

$$R = v^2/p = 127^2/100\,\Omega \approx 161\,\Omega$$

Question 2 [Voltage divider]
Regarding the following voltage-divider circuit



a) [2 marks] Show that if $R_1 = R_3 = R\Omega$ and $R_2 = 2R\Omega$ then $v_A - v_B = v_C - v_D = 2.5V$ and $v_B - v_C = 5V$.

The current flowing through all three resistors is

$$i = 10/(R_1 + R_2 + R_3) = 10/4R = (2.5/R)A$$

Therefore the voltages across the resistors can be computed

$$\begin{aligned} v_A - v_B &= R_1 i = R(2.5/R) = 2.5V \\ v_B - v_C &= R_2 i = 2R(2.5/R) = 5.0V \\ v_C - v_D &= R_3 i = R(2.5/R) = 2.5V \end{aligned}$$

b) [2 marks] Use Thevenin's equivalence to show that at terminals B and C the equivalent resistance is $R\Omega$. Draw the equivalent Thevenin circuit.

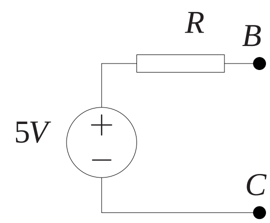
First show that the open circuit voltage $v_T = v_B - v_C = 5.0V$ as computed in item a).
Then compute the short circuit current i_N which is the current flowing through resistors R_1 and R_3 when R_2 is short circuited

$$i_N = 10/(R_1 + R_3) = 10/2R = (5.0/R)A$$

Therefore the equivalent resistance is

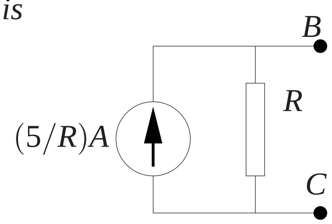
$$R_{EQ} = v_T/i_N = 5.0/(5.0/R) = R\Omega$$

The equivalent Thevenin circuit is

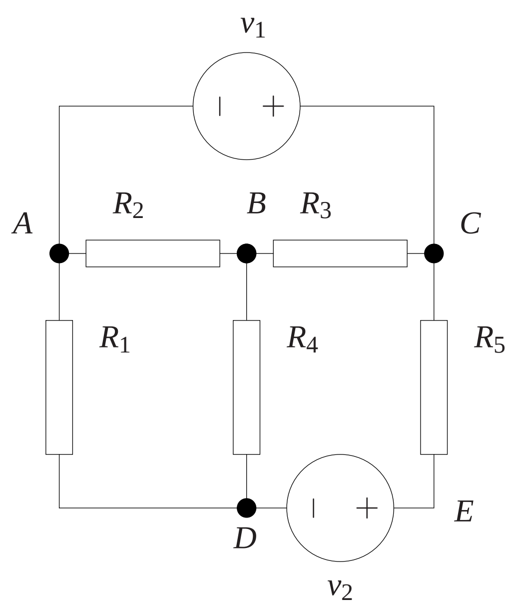


c) [1 mark] Draw the Norton equivalent circuit as seen from terminals B and C.

The equivalent Norton circuit is

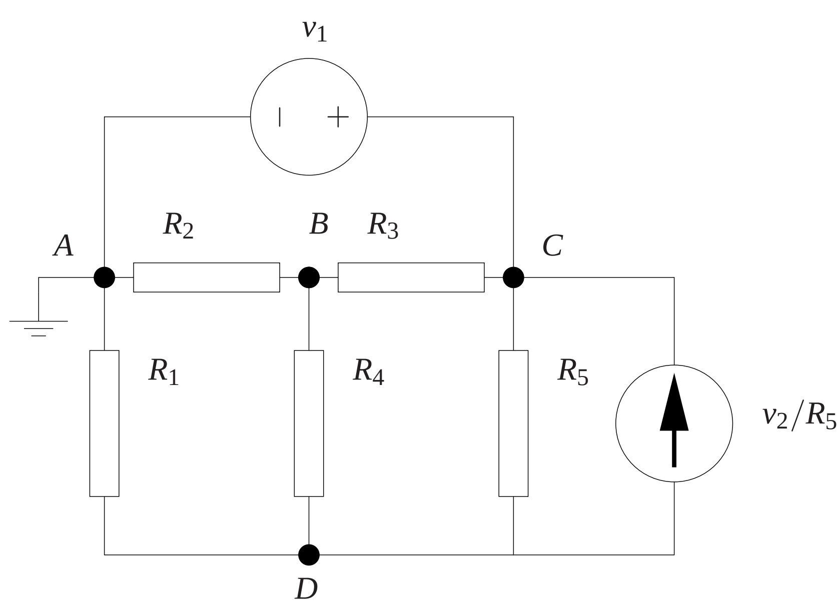


Question 3 [Node-voltage analysis]
Regarding the following circuit



a) [2 marks] Transform the series connection of the voltage source v_2 and the resistor R_5 into an equivalent current source in parallel with a resistance and draw the resulting circuit.

The resulting circuit is



b) [3 marks] Formulate node-voltage equations for the resulting circuit. Remember to properly choose the ground node! Clearly state the unknowns and the equations to be solved.

First note that there is still one source of voltage that cannot be transformed into a source of *current. In order to handle that choose the ground to be node A, as indicated in the figure above. C is an alternative.*

Now write the node-voltage equations by inspection

$$\begin{bmatrix} \frac{1}{R_2}+\frac{1}{R_3}+\frac{1}{R_4} & -\frac{1}{R_3} & -\frac{1}{R_4} \\ -\frac{1}{R_3} & \frac{1}{R_3}+\frac{1}{R_5} & -\frac{1}{R_5} \\ -\frac{1}{R_4} & -\frac{1}{R_5} & \frac{1}{R_1}+\frac{1}{R_4}+\frac{1}{R_5} \end{bmatrix} \begin{pmatrix} v_B \\ v_C \\ v_D \end{pmatrix} = \begin{pmatrix} 0 \\ i_{v1}+\frac{v_2}{R_5} \\ -\frac{v_2}{R_5} \end{pmatrix}$$

where i_{v_1} is the current on the voltage source v_1.

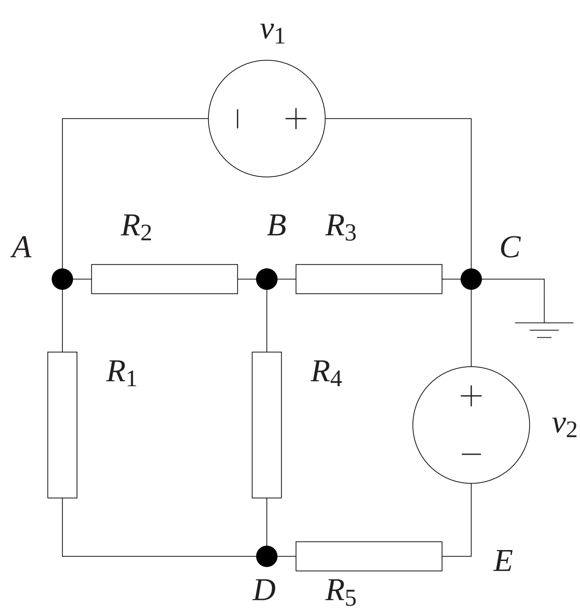
Having set A as ground, $v_C = v_1$ is now known and can be eliminated from the vector of unknowns. Furthermore, the presence of the current i_{v_1} on the right hand side of the second equation means that this equation can be solved after v_B and v_D are determined. This results in the smaller set of equations

$$\begin{bmatrix} \frac{1}{R_2}+\frac{1}{R_3}+\frac{1}{R_4} & -\frac{1}{R_4} \\ -\frac{1}{R_4} & \frac{1}{R_1}+\frac{1}{R_4}+\frac{1}{R_5} \end{bmatrix} \begin{pmatrix} v_B \\ v_D \end{pmatrix} = \begin{pmatrix} \frac{v_1}{R_3} \\ \frac{v_1}{R_5}-\frac{v_2}{R_5} \end{pmatrix}$$

to be solved for v_B and v_D.

c) [BONUS - 1 mark] Can you formulate node-voltage equations for the above circuit without transforming the voltage source v_2 into an equivalent current source?

One way is using supernodes. We will do that later. Another way is noting that we can commute v_2 with R_2 to produce the following equivalent circuit



Now write the node-voltage equations by inspection. Because $v_A = -v_1$ and $v_E = -v_2$ are known we need only to write the current equations at nodes B and D. This results in

$$\begin{bmatrix} -\frac{1}{R_2} & \frac{1}{R_2}+\frac{1}{R_3}+\frac{1}{R_4} & -\frac{1}{R_4} & 0 \\ -\frac{1}{R_1} & -\frac{1}{R_4} & \frac{1}{R_1}+\frac{1}{R_4}+\frac{1}{R_5} & -\frac{1}{R_5} \end{bmatrix} \begin{pmatrix} v_A \\ v_B \\ v_D \\ v_E \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Having set C as ground, $v_A = -v_1$ and $v_E = -v_2$ are now known and can be eliminated from the vector of unknowns. This results in the smaller set of equations

$$\begin{bmatrix} \frac{1}{R_2}+\frac{1}{R_3}+\frac{1}{R_4} & -\frac{1}{R_4} \\ -\frac{1}{R_4} & \frac{1}{R_1}+\frac{1}{R_4}+\frac{1}{R_5} \end{bmatrix} \begin{pmatrix} v_B \\ v_D \end{pmatrix} = \begin{pmatrix} -\frac{v_1}{R_2} \\ -\frac{v_1}{R_1}-\frac{v_2}{R_5} \end{pmatrix}$$

on the unknowns v_B and v_D.

Now let us do it the alternative way: creating a supernode D-E on the original diagram. Setting the ground to node A we write node-voltage equations by inspection on the nodes B, C, D and E

$$\begin{bmatrix} \frac{1}{R_2}+\frac{1}{R_3}+\frac{1}{R_4} & -\frac{1}{R_3} & -\frac{1}{R_4} & 0 \\ -\frac{1}{R_3} & \frac{1}{R_3}+\frac{1}{R_5} & 0 & -\frac{1}{R_5} \\ -\frac{1}{R_4} & 0 & \frac{1}{R_1}+\frac{1}{R_4} & 0 \\ 0 & -\frac{1}{R_5} & 0 & \frac{1}{R_5} \end{bmatrix} \begin{pmatrix} v_B \\ v_C \\ v_D \\ v_E \end{pmatrix} = \begin{pmatrix} 0 \\ i_{v1} \\ -i_{v2} \\ i_{v2} \end{pmatrix}$$

where i_{v_1} is the current on the voltage source v_1 and i_{v_2} is the current on the voltage source v_2.

The sum of the third and fourth equation provide the current balance for the supernode D-E. The second equation can be eliminated because of the presence of i_{v_1}. That reduces the problem to

$$\begin{bmatrix} \frac{1}{R_2}+\frac{1}{R_3}+\frac{1}{R_4} & -\frac{1}{R_3} & -\frac{1}{R_4} & 0 \\ -\frac{1}{R_4} & -\frac{1}{R_5} & \frac{1}{R_1}+\frac{1}{R_4} & \frac{1}{R_5} \end{bmatrix} \begin{pmatrix} v_B \\ v_C \\ v_D \\ v_E \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Having set A as ground, $v_C = v_1$ is now known and can be eliminated from the vector of unknowns. We can also eliminate v_E by noticing that $v_E = v_D + v_2$.

$$\begin{bmatrix} \frac{1}{R_2}+\frac{1}{R_3}+\frac{1}{R_4} & -\frac{1}{R_4} \\ -\frac{1}{R_4} & \frac{1}{R_1}+\frac{1}{R_4}+\frac{1}{R_5} \end{bmatrix} \begin{pmatrix} v_B \\ v_D \end{pmatrix} = \begin{pmatrix} \frac{v_1}{R_3} \\ \frac{v_1}{R_5}-\frac{v_2}{R_5} \end{pmatrix}.$$

As a final remark, note that, at first, the two answers do not seem to be the same, as the right hand sides are different. However, recalling that C is the ground in the first approach, we see that v_B and v_D cannot be the same. Indeed, in the first approach what we have implicitely computed is

$$\begin{bmatrix} \frac{1}{R_2}+\frac{1}{R_3}+\frac{1}{R_4} & -\frac{1}{R_4} \\ -\frac{1}{R_4} & \frac{1}{R_1}+\frac{1}{R_4}+\frac{1}{R_5} \end{bmatrix} \begin{pmatrix} v_B - v_C \\ v_D - v_C \end{pmatrix} = \begin{pmatrix} -\frac{v_1}{R_2} \\ -\frac{v_1}{R_1}-\frac{v_2}{R_5} \end{pmatrix}$$

Now using the fact that $v_C = v_A + v_1$ so that

$$\begin{aligned} \begin{bmatrix} \frac{1}{R_2}+\frac{1}{R_3}+\frac{1}{R_4} & -\frac{1}{R_4} \\ -\frac{1}{R_4} & \frac{1}{R_1}+\frac{1}{R_4}+\frac{1}{R_5} \end{bmatrix} \begin{pmatrix} v_B - v_A \\ v_D - v_A \end{pmatrix} &= \begin{pmatrix} -\frac{v_1}{R_2} \\ -\frac{v_1}{R_1}-\frac{v_2}{R_5} \end{pmatrix} + \begin{pmatrix} \frac{1}{R_2}+\frac{1}{R_3} \\ \frac{1}{R_1}+\frac{1}{R_5} \end{pmatrix} v_1 \\ &= \begin{pmatrix} \frac{v_1}{R_3} \\ \frac{v_1}{R_5}-\frac{v_2}{R_5} \end{pmatrix} \end{aligned}$$

which proves that the two answers are indeed the same, if measured relative to the same reference.