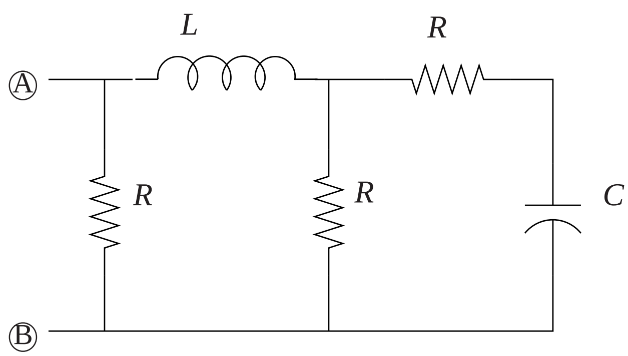
MAE 140 – Linear Circuits – Summer 2007
Final Solution

NOTE: The items d) and e) of Question 4 gave you bonus marks.

Question 1 [Equivalent Circiuts]
[4 marks] Find the equivalent impedance between terminals Ⓐ and Ⓑ in the following circuits.



$$Z(s) = R//[sL + R//(R + 1/sC)]$$

Let's compute one term at a time. First

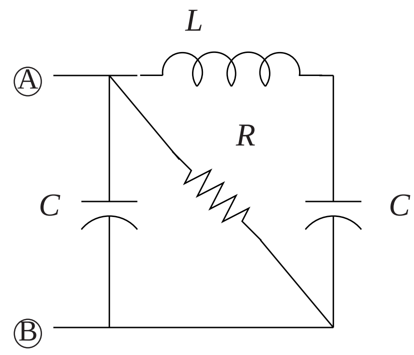
$$R//(R+1/sC) = \frac{R(R+1/sC)}{R+(R+1/sC)} = \frac{R(sRC+1)}{sRC+(sRC+1)},$$

followed by

$$sL + R//(R+1/sC) = sL + \frac{R(sRC+1)}{sRC+(sRC+1)} = \frac{s^2RLC+(sL+R)(sRC+1)}{sRC+(sRC+1)}.$$

Finally

$$\begin{aligned} Z(s) &= R//[sL + R//(R+1/sC)] \\ &= \frac{\frac{s^2R^2LC+R(sL+R)(sRC+1)}{sRC+(sRC+1)}}{R+\frac{s^2RLC+(sL+R)(sRC+1)}{sRC+(sRC+1)}} = \frac{s^2R^2LC+R(sL+R)(sRC+1)}{sRC(sL+R)+(sL+2R)(sRC+1)}. \end{aligned}$$

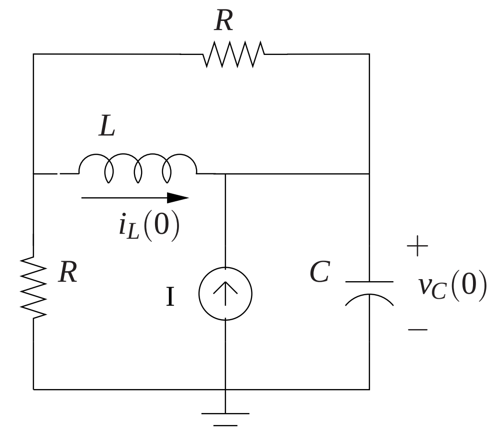


$$Z(s) = 1/sC//R//(sL + 1/sC)$$

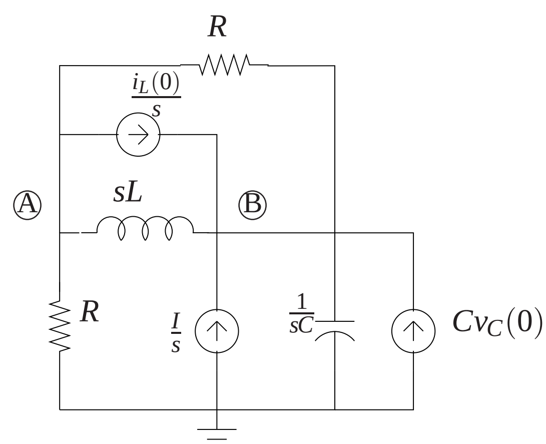
so all there is to do is to compute the parallel association

$$Z(s) = \frac{1}{sC + \dfrac{1}{R} + \dfrac{sC}{s^2LC+1}} = \frac{R(s^2LC+1)}{sCR + (sCR+1)(s^2LC+1)}.$$

Question 2 [Nodal Analysis in the s-Domain]
[6 marks] Transform the following circuit to the s-domain and formulate node-voltage equations. Assume initial conditions and reference ground as indicated in the figure. The current source is constant.



Transform to the s-domain using sources of current for initial conditions, as we will need to write node-voltage equations.

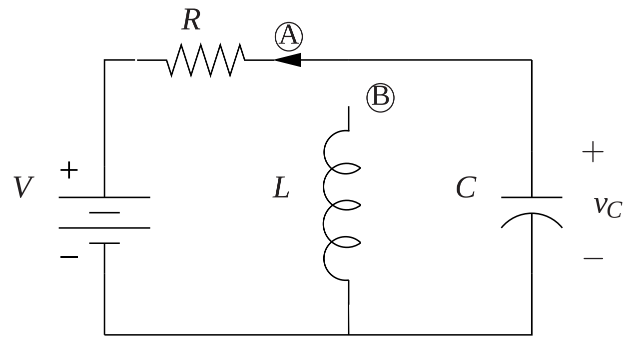


Now write node-voltage equations by inspection:

$$\begin{bmatrix} \frac{1}{sL}+\frac{2}{R} & -\frac{1}{sL}-\frac{1}{R} \\ -\frac{1}{sL}-\frac{1}{R} & \frac{1}{sL}+sC+\frac{1}{R} \end{bmatrix} \begin{pmatrix} V_A(s) \\ V_B(S) \end{pmatrix} = \begin{pmatrix} -\frac{i_L(0)}{s} \\ \frac{I+i_L(O)}{s}+Cv_C(0) \end{pmatrix}.$$

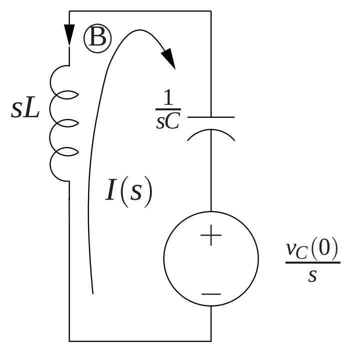
Question 3 [Transient Analysis in the s-Domain]

[6 marks] The switch in the next circuit has been left in position Ⓐ for a long time and is moved to position Ⓑ at $t = 0$. Find $v_c(t)$ for $t \geq 0$. The voltage source is constant.



The first step is to determine $v_C(0)$. This is obtained by noticing that if the switch is on Ⓐ for a long time then the current in C, and consequently in R, should be zero, therefore, $v_C(0) = V$.

Now transform the circuit after the switch is moved to Ⓑ to the s-domain as in the following diagram.



Mesh analysis applied to the diagram provides

$$(sL + 1/sC)\,I(s) = -\frac{v_C(0)}{s} \qquad \Rightarrow \qquad I(s) = -\frac{v_C(0)}{s(sL + 1/sC)} = -\frac{CV}{s^2LC + 1}$$

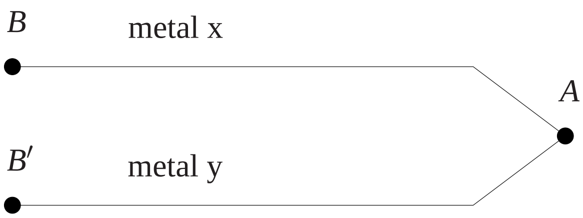
Noticing that

$$v_C(s) = \frac{v_C(0)}{s} + \frac{1}{sC} I(s) = \left(1 - \frac{1}{s^2LC + 1}\right)\frac{V}{s} = \frac{sLC}{s^2LC + 1} V = \frac{s}{s^2 + 1/LC} V$$

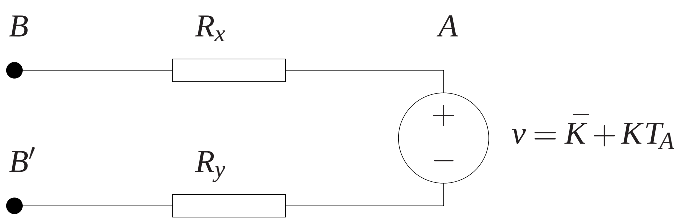
Therefore, applying the Laplace inverse we have

$$v_c(t) = V\mathscr{L}^{-1}\left\{\frac{s}{s^2 + 1/LC}\right\} = V\cos(\omega t)\,u(t), \qquad \omega := \sqrt{\frac{1}{LC}}.$$

Question 4 [Circuit Variables and OpAmp Circuit Design]
When two *different* metal wires are placed in contact (creating a junction) a voltage appears that is proportional to the junction temperature and the material properties. A pair of wires made with different materials connected at one end as in the next figure is known as a *thermocouple*, and is a very popular temperature sensor. No voltage appears on junctions made of same materials because of temperature. The points B and B' are at the same temperature T_B.

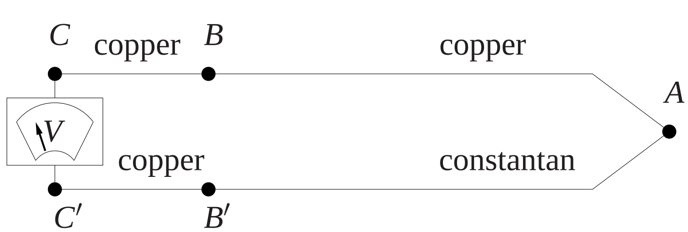


A good model for the thermocouple junction is as a voltage source with voltage $v = \bar{K} + KT$, where $\bar{K}$ and K are constants that depend only on the material used in the junction and T is the junction temperature. A simple circuit model for the above thermocouple is given in the next diagram, where R_x and R_y represent the resistance of the wires, which are essentially functions of the cross section area and length of the thermocouple. It is fair to assume that $R_x \approx R_y$.

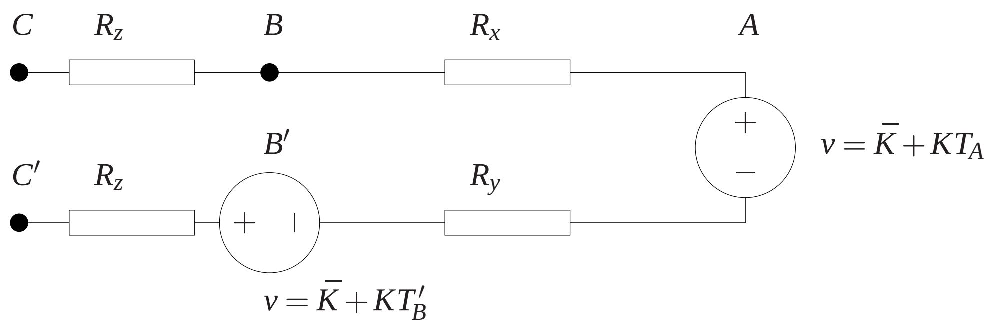


A thermocouple made with 'metal x' being copper and 'metal y' being constantan can measure temperatures in the range -200 ^{o}C to 350 ^{o}C with $K = 43\mu$ V/^{o}C. The voltage v is measured from the copper terminal $(+)$ to the constantan terminal $(-)$. As you will see soon, the value of $\bar{K}$ is not important.

a) [3 marks] A friend of yours suggested that you can measure the temperature of point A (T_A) by simply connecting a voltmeter with copper leads to the points B and B' and measure the resulting voltage in C and C' (internal to the voltmeter), as in the next figure. The points B and B' are at the same temperature T_B. The points C and C' are at the same temperature T_C. Draw the circuit diagram corresponding to this setup and show that he/she is not correct: this setup can only measure $V_C - V_{C'} = K(T_A - T_B)$.
(Hint: remember that a voltage appear on all junctions made with different materials!)



Because the junction B' has two different metals the circuit diagram of the above setup is as follows:



Because no current flows into the voltmeter, then

$$V_C - V'_C = (\bar{K} + KT_A) - (\bar{K} + KT'_B) = K(T_A - T'_B) = K(T_A - T_B)$$

as stated.

One way of overcomming the above problem is to let the temperature T_B be known. A popular approach is to have the junction B' be immersed in a bath of water and ice, in which the temperature is exactly 0.01 ^{o}C (known as the triple point of water) so that $T_A = T_B + (V_C - V_{C'})/K \approx (V_C - V_{C'})/K$.

b) [4 marks] Assume that T_B is in a cold bath at the triple point of water (assume $T_B \approx 0$) and design an OpAmp circuit to be connected at C-C' that outputs a voltage $v_0 = \alpha T_A$, where $\alpha = 10$ m V/^{o}C. Note that this circuit should makes the measurement independent of the wire resistance. If the OpAmp is powered with $+10V$ and $-10V$ what is the temperature range that you can measure accurately with your circuit?

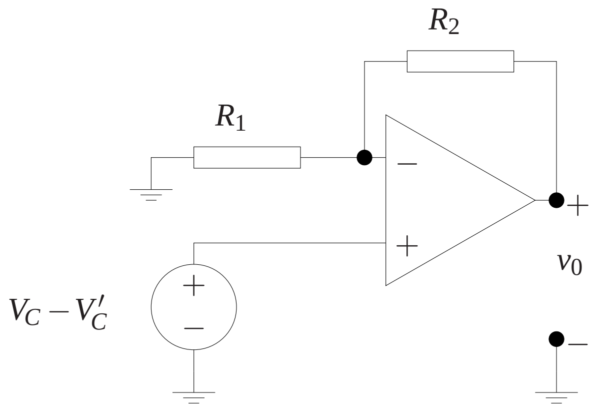
Because $T_A \approx (V_C - V'_C)/K$ our circuit have to implement the function

$$\begin{aligned} v_0 &= \alpha T_A \\ &\approx (\alpha/K)(V_C - V'_C) \end{aligned}$$

with gain

$$\frac{\alpha}{K} = \frac{10 \times 10^{-3} V/^oC}{43 \times 10^{-6} V/^oC} = \frac{10}{43} \times 10^3 \approx 232.$$

One possible answer is to use a non-inverting amplifier setup



with gain

$$\frac{R_2+R_1}{R_1} = 1 + \frac{R_2}{R_1} = 232.$$

This configuration has a high input impedance so the effect of the ohmic wire resistances is minized. Possible (unrealistic) choices for R_1 and R_2 that could be used are

$$R_1 = 1K\Omega, \qquad R_2 = 231K\Omega.$$

We should be able to read temperatures while the OpAmp stays in the linear range. So we look for the saturation points:

$$10\times10^{-3}T_{\text{low}} = -10 \quad\Rightarrow\quad T_{\text{low}} = -1000\,^{\text{o}}C,$$
$$10\times10^{-3}T_{\text{high}} = 10 \quad\Rightarrow\quad T_{\text{high}} = +1000\,^{\text{o}}C.$$

As the thermocouple is accurate in the range $[-200,350]\,^{\text{o}}C$ we should be able to read accurately the entire scale of the thermocouple. Indeed, we could have used a higher gain to enhance the circuit resolution, perhaps using two stages of amplification.

Another way of overcomming the temperature reference problem is to directly measure the temperature T_B. The justification for this is that T_B is the temperature of a controlled environment, say your workbench, while T_A may be an extreme temperature you're trying to measure. Therefore, you could use a temperature sensor to measure T_B that is less expensive or perhaps acurate only on ambient temperature. One such device is called a *termistor*, which is a resistor whose resitance varies with the temperature. Termistors are typically accurate and approximately linear from 0 $^{\text{o}}$C to a dozen degrees above ambient temperature.

c) [2 marks] You have a linear termistor with a resistance of $30K\Omega$ at 0 $^{\text{o}}$C and a resistance of $10K\Omega$ at 20 $^{\text{o}}$C. Show that the relationship between the termistor resistance (R_T) and the termistor temperature (T_B) is

$$R_T = (30 - T_B)\times 10^3\,\Omega.$$

Up to what temperature do you think this termistor is acurate (or at least linear)? Why?

Because the termistor is linear it should satisfy

$$R_T = aT_B + b$$

for some a and b. Evaluating R_T at $T_B = 0\ ^{o}C$ and $T_B = 20\ ^{o}C$ yields

$$a \times 0 + b = b = 30 \times 10^3 \Omega, \qquad a \times 20 + b = 10 \times 10^3 \Omega$$

from where

$$b = 30 \times 10^3$$
$$a = (10 \times 10^3 - b)/20 = (10 \times 10^3 - 30 \times 10^3)/20 = -1 \times 10^3 \Omega/^{o}C.$$

Hence

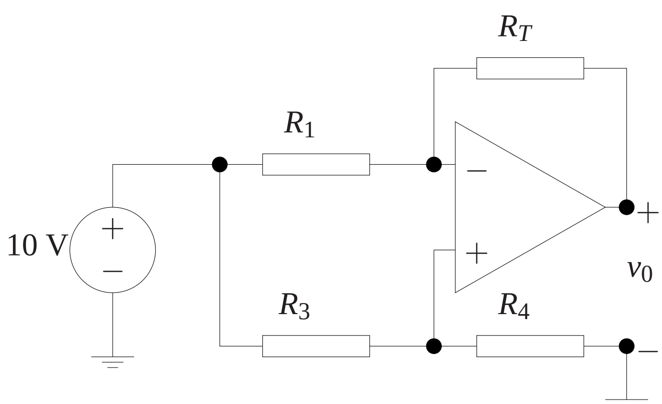
$$R_T = (30 - T_B) \times 10^3\, \Omega.$$

You should start being suspicious about this termistor model when it reaches temperatures for which R_T becomes near zero, that is, near

$$(30 - T_B) \times 10^3 = 0 \qquad \Rightarrow \qquad T_B = 30^{o}C.$$

You should then worry about using it in a hot day or improve your linear assumption :).

d) [Bonus: 4 marks] Using the above relationship between the termistor resistance and temperature find values for the components R_1, R_3 and R_4 so that the following circuit produces $v_0 = \alpha T_B$, where $\alpha =$ 10 mV/^{o}C and T_B is the temperature of the termistor and the junction B'.



First recognize that the above circuit is a differential amplifier where

$$v_0 = K_1 v_1 + K_2 v_2, \qquad K_1 = -\frac{R_T}{R_1}, \qquad K_2 = \frac{R_T + R_1}{R_1}\frac{R_4}{R_3 + R_4} = \left(1 + \frac{R_T}{R_1}\right)\frac{R_4}{R_3 + R_4}$$

and $v_1 = v_2 = 10V$. Using this fact

$$\begin{aligned} v_0 &= 10(K_1 + K_2) \\ &= 10\left(\frac{R_1+R_T}{R_1}\frac{R_4}{R_3+R_4} - \frac{R_T}{R_1}\right) \\ &= \frac{10}{R_1}\left(\frac{R_4(R_1+R_T)}{R_3+R_4} - R_T\right) \\ &= \frac{10}{R_1(R_3+R_4)}(R_1R_4 - R_3R_T) \end{aligned}$$

Now substitute for $R_T = R_0 - \beta T_B$, where $R_0 = 30\times 10^3\Omega$ and $\beta = 10^3$

$$\begin{aligned} v_0 &= \frac{10}{R_1(R_3+R_4)}(R_1R_4 - R_3R_0 + \beta R_3 T_B) \\ &= \frac{10}{R_1(R_3+R_4)}(R_1R_4 - R_3R_0) + \frac{10\beta R_3}{R_1(R_3+R_4)}T_B. \end{aligned}$$

For $v_0 = \alpha T_B$ we need to chose

$$\frac{R_3}{R_4} = \frac{R_1}{R_0}, \qquad \frac{10\beta R_3}{R_1(R_3+R_4)} = \frac{10\beta}{R_1\left(1+\frac{R_4}{R_3}\right)} = \alpha$$

This fixes the choice of R_1 since

$$R_1 = 10\beta\alpha^{-1} - R_0 = 10\times 10^3 \times 10^2 - 30\times 10^3 = (1000-30)K\Omega = 970K\Omega$$

Possible choices of R_3 and R_4 are

$$R_3 = R_1 = 970K\Omega, \qquad R_4 = R_0 = 30K\Omega, \qquad \frac{R_3}{R_4} = \frac{R_1}{R_0} = \frac{970\times 10^3}{30\times 10^3} \approx 32$$

e) [Bonus: 4 marks] Design an OpAmp circuit that has as output voltage $v_0 = \alpha(T_A - T_B)$, where $\alpha = 10$ mV/^{o}C and T_B is measured using the termistor as in item d).
(Hint: use the circuit you designed in item d)!)

The simplest solution is to reuse the circuits developed in items b) and d) through a differential amplifier configuration with

$$v_0 = K_1v_1 + K_2v_2, \qquad K_1 = -\frac{R_2}{R_1}, \qquad K_2 = \frac{R_2+R_1}{R_1}\frac{R_4}{R_3+R_4}$$

with $v_1 = \alpha T_B$ as in item d) and $v_2 = \alpha T_A$ as in item b). This requires

$$K_1 = -\frac{R_2}{R_1} = -1, \qquad K_2 = 2\frac{R_4}{R_3+R_4} = 1$$

Possible choices of R_1, R_2, R_3 and R_4 are

$$R_1 = R_2 = R_3 = R_4 = 100K\Omega.$$

The final circuit diagram is as follows.