MAE140 - Linear Circuits - Fall 09
Final, December 7

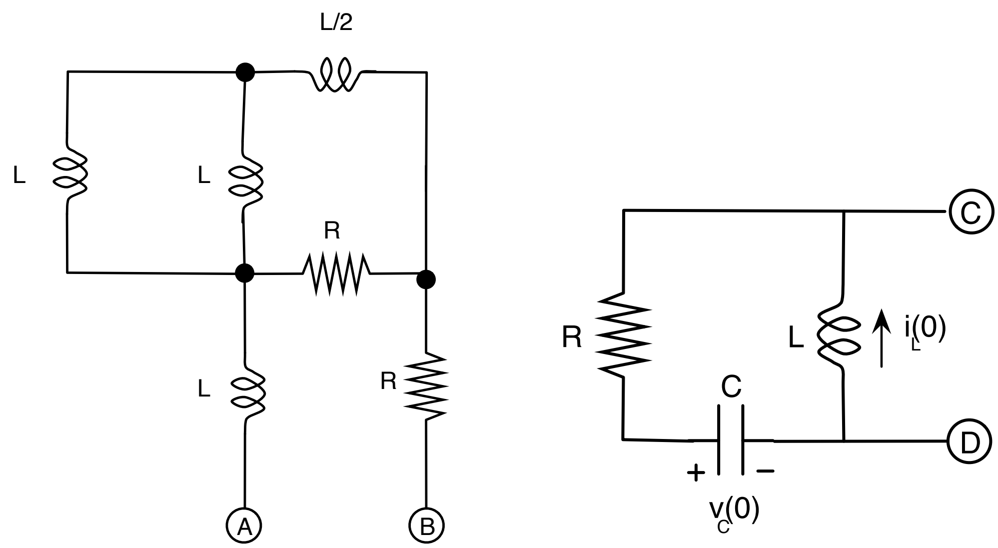
Instructions

(i) This exam is open book. You may use whatever written materials you choose, including your class notes and textbook. You may use a hand calculator with no communication capabilities

(ii) You have 170 minutes

(iii) Do not forget to write your name, student number, and instructor

(iv) On the questions for which we have given the answers, please provide detailed derivations.



(a) Question 1, Part I (b) Question 1, Part II

Figure 1: Circuit for Question 1.

1. **Equivalent Circuits**

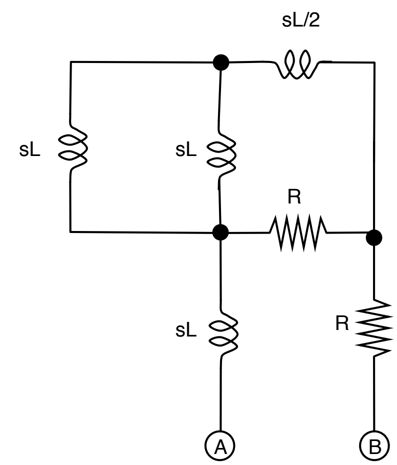
Part I: [5 points] Assuming zero initial conditions, find the impedance equivalent to the circuit in Figure 1(a) as seen from terminals A and B. The answer should be given as a ratio of two polynomials.

Part II: [5 points] Assuming that the initial conditions of the inductor and capacitor are as indicated in the diagram, redraw the circuit shown in Figure 1(b) in the s-domain. Then use source transformations to find the s-domain Norton equivalent of this circuit as seen from terminals C and D. (**Hint:** Use an equivalent model for the inductor in which the initial condition appears as a current source, and an equivalent model for the capacitor in which the initial condition appears as a voltage source)

Solution: Part I:

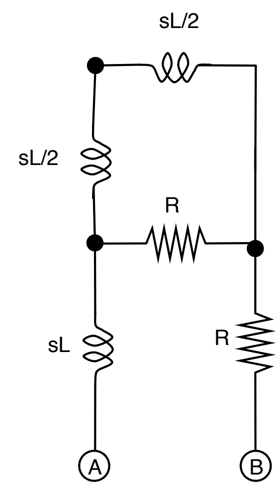
Since all initial conditions are zero, it is easy to transform the circuit to the s-domain.

[1 point]



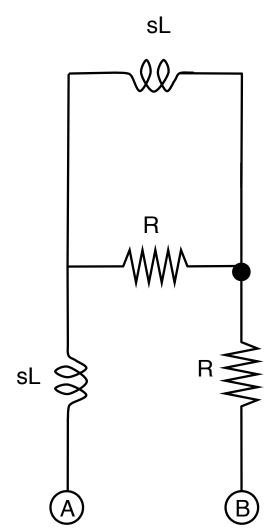
Now, we combine the two impedances (inductors) in parallel to arrive to the circuit on the right.

[1 point]



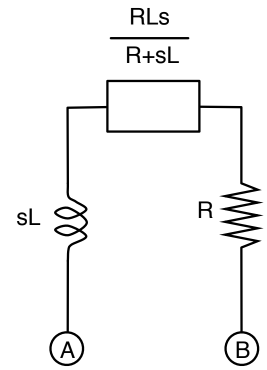
And now the two impedances in series to arrive to the circuit on the right.

[1 point]



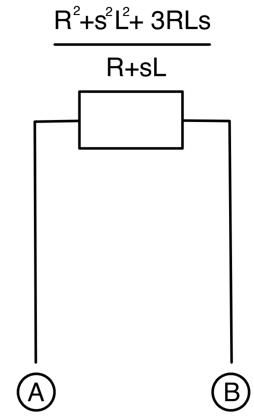
Now again the two impedances in parallel to arrive to the circuit on the right.

[1 point]



Finally, we arrive at the equivalent impedance by merging together all the impedances in series

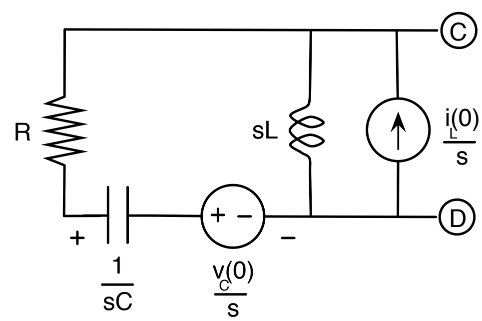
[1 point]



Part II:

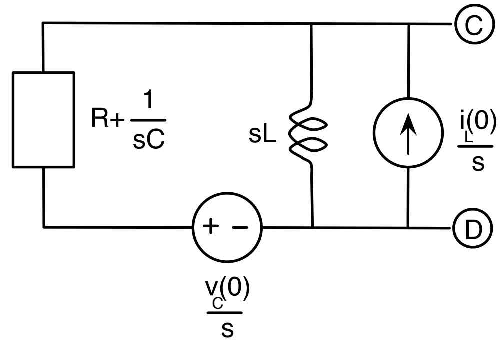
We begin by transforming the circuit to the s-domain taking good care of the initial conditions using the hint.

[1 point for capacitor transformation; 1 point for inductor transformation]



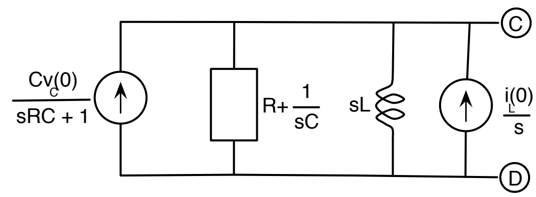
Next, we combine the resistor and the capacitor in series

[1 point]



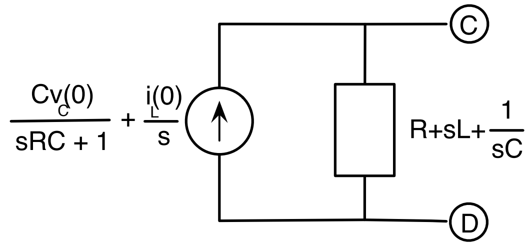
Now, we do a source transformation with the voltage source and the impedance in series

[1 point]



Combining everything together, we get the Norton equivalent

[1 point]



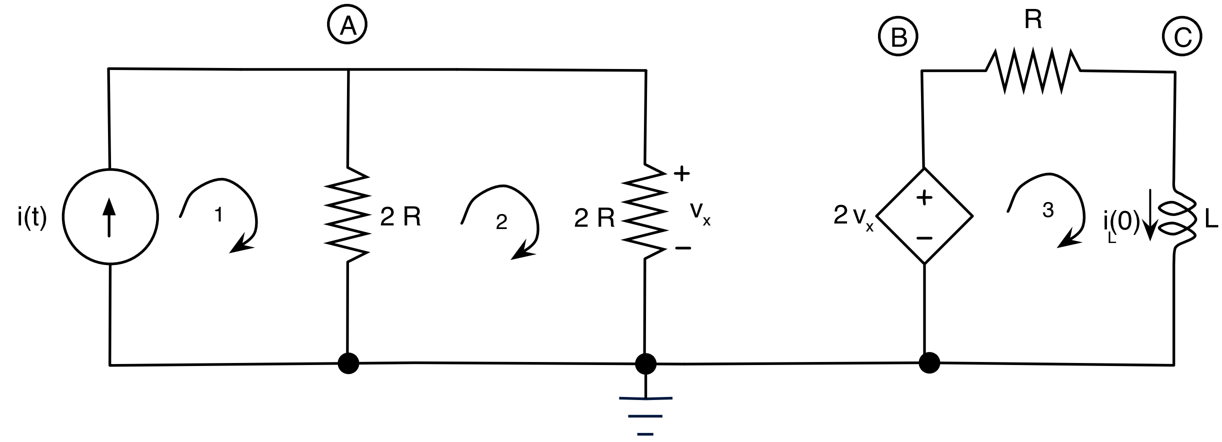


Figure 2: Nodal and Mesh Analysis Circuit

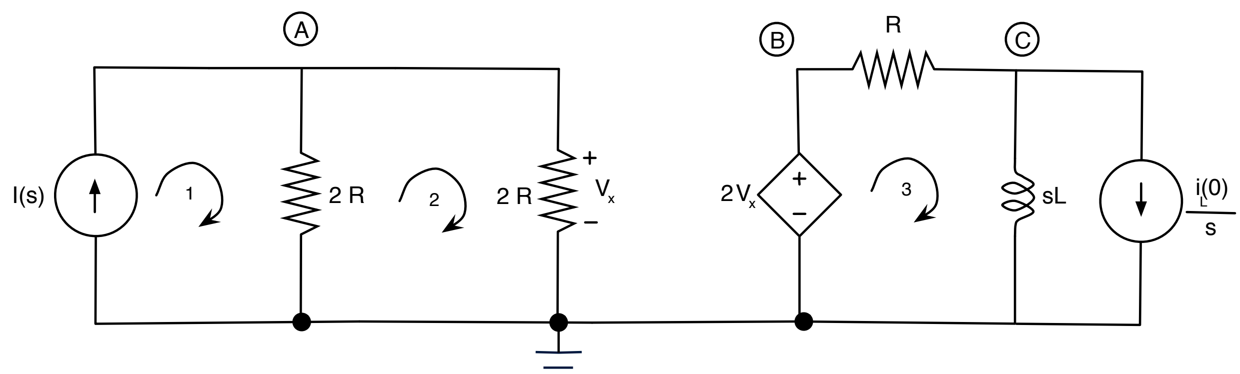
2. **Nodal and Mesh Analysis**

Part I: [5 points] Formulate node-voltage equations in the s-domain for the circuit in Figure 2. Use the reference node and other labels as shown in the figure. Do not assume zero initial conditions! Transform the initial condition on the inductor into a current source.

Part II: [5 points] Formulate mesh-current equations in the s-domain for the circuit in Figure 2. Use the mesh currents shown in the figure. Do not assume zero initial conditions! Transform the initial condition on the inductor into a voltage source.

Solution:

Part I:



In the circuit above, we have transformed the circuit to the s-domain, taking good care of respecting the orientation of the current.

[1 point]

The dependent voltage source poses a problem for nodal analysis, but we can easily take care of it thanks to the location of the ground node,

$$V_B = 2V_x$$

[1 point]

Therefore, we only to need to write equations for nodes A and C. For node A, we have

$$.5GV_A + .5GV_A - I(s) = 0$$

where $G = 1/R$.

[1 point]

For node C, we have

$$G(V_C - V_B) + \frac{1}{sL}V_C + \frac{i_L(0)}{s} = 0$$

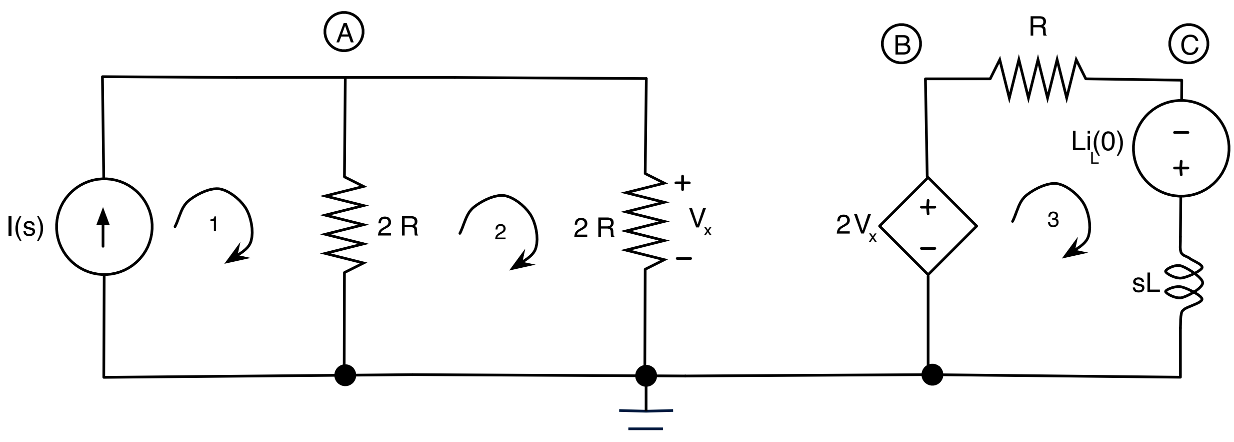
[1 point]

Therefore, we have a total of 3 equations with 4 unknowns, because of the presence of the dependent source. The additional equation that we need is simply

$$V_A = V_x$$

[1 point]

Part II:



In the circuit above, we have transformed the circuit to the s-domain, taking good care of respecting the polarity.

[1 point]

The current source on the left poses a problem for mesh analysis, but we can easily take care of it by realizing that it only belongs to one mesh, and therefore

$$I_1 = I(s)$$

[1 point]

Therefore, we only need to write equations for meshes 2 and 3. For mesh 2, we have

$$2RI_2 + 2R(I_2 - I_1) = 0$$

[1 point]

For mesh 3, we have

$$RI_3 + sLI_3 = Li_L(0) + 2V_x$$

[1 point]

Finally, note that

$$V_x = 2RI_2$$

[1 point]

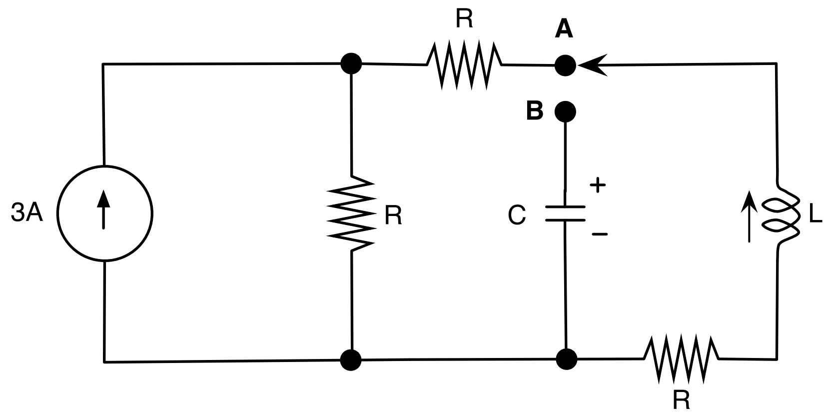


Figure 3: RCL circuit for Laplace Analysis

3. **Laplace Domain Circuit Analysis**

Part I: [3 points] Consider the circuit depicted in Figure 3. The current source is constant. The switch is kept in position **A** for a very long time. At $t = 0$ it is moved to position **B**. Show that the initial capacitor voltage and inductor currents are given by

$$v_C(0^-) = 0V, \quad i_L(0^-) = -1A.$$

[Show your work]

Part II: [2 points] Use these initial conditions to transform the circuit into the s-domain for $t \geq 0$. Use equivalent models for the capacitor and the inductor in which the initial conditions appear as current sources.

[Show your work]

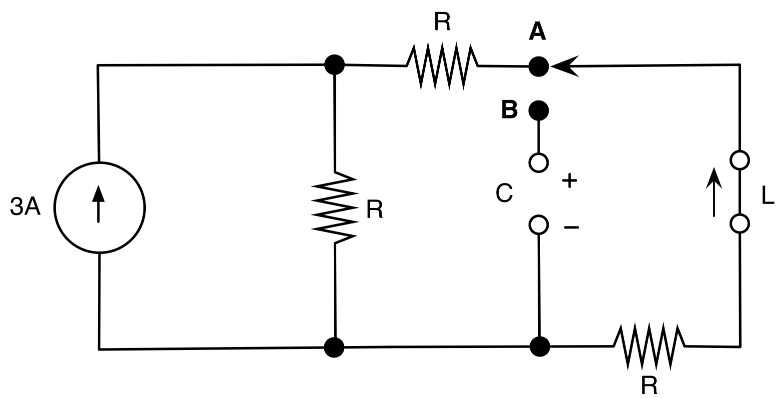
Part III: [5 points] Use domain circuit analysis and inverse Laplace transforms to show that the inductor current when $C = 1F$, $L = 1H$, and $R = 2\Omega$ is

$$i_L(t) = (-e^{-t} + te^{-t})u(t).$$

Solution:

Part I:

We substitute the inductor by a short circuit and the capacitor by an open circuit to find their initial conditions.



[1 point]

Then it is clear from the circuit that

$$v_C(0^-) = 0.$$

[1 point]

Using current division, we find that the initial condition for the inductor is

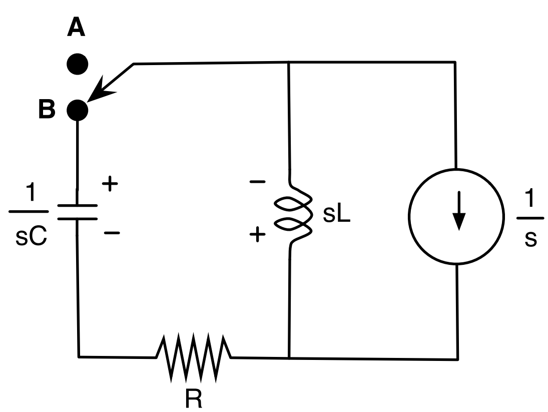
$$i_L(0^-) = -\frac{\frac{1}{2R}}{\frac{1}{2R} + \frac{1}{R}} 3 = -1\,A$$

[1 point]

Part II:

Since there is no initial voltage across the capacitor, we do not need to add an independent current source for it. **[1 point]**

We add one current source in parallel for the inductor to take care of its initial condition.



[1 point]

Part III:

We can find the current through the impedance sL by current division as

$$I_{sL} = \frac{1/sL}{1/sL + sC/(sRC+1)} \frac{1}{s}$$

[1 point]

The transform of the current through the inductor L is then

$$I_L(s) = I_{sL} - \frac{1}{s} = \frac{sRC+1}{(s^2CL + sRC + 1)s} - \frac{1}{s} = \frac{2s+1}{(s^2+2s+1)s} - \frac{1}{s}$$

[1 point]

To find the inductor current, we need to compute the inverse Laplace transform. Using partial fractions, we set

$$I_L(s) = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+1)^2} - \frac{1}{s}$$

[1 point]

You can use any method you like to find the constants A, B, and C. We use here the cover-up method

$$\begin{aligned} A &= \lim_{s\to 0} sI_{sL}(s) = 1 \\ C &= \lim_{s\to -1} (s+1)^2 I_{sL}(s) = 1 \\ B &= \lim_{s\to -1} \frac{d}{ds}\left((s+1)^2 I_{sL}(s)\right) = -1 \end{aligned}$$

Therefore, we have

$$I_L(s) = \frac{1}{s} - \frac{1}{s+1} + \frac{1}{(s+1)^2} - \frac{1}{s} = -\frac{1}{s+1} + \frac{1}{(s+1)^2}$$

[1 point]

The inductor current is then

$$i_L(t) = (-e^{-t} + te^{-t})u(t)$$

[1 point]

4. **Frequency Response**

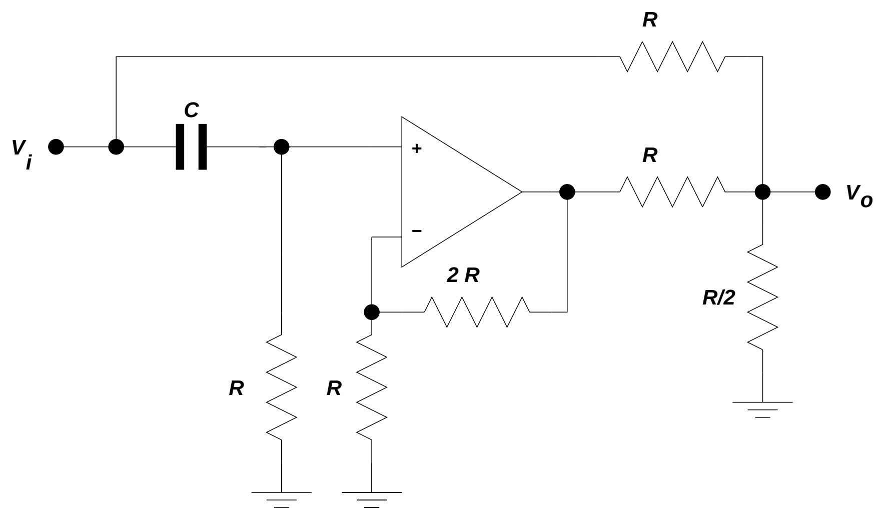


Figure 4: Circuit for Question 4.

Part I [4 points] Show that the transfer function from $V_i(s)$ to $V_o(s)$ for the circuit in Figure 4 is given by

$$T(s) = \frac{V_o(s)}{V_i(s)} = \frac{s + 1/(4RC)}{s + 1/(RC)}$$

Show your work.

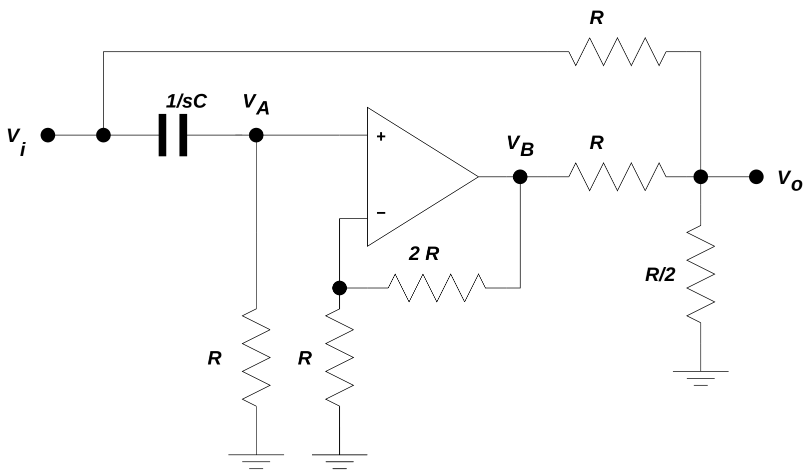
Part II [4 points] Let $R = 10$ KΩ, $C = 100$ nF and compute the magnitude and phase of $T(s)$ at $s = j\omega$ where $\omega = \{0, 250, 1000, \infty\}$ rad/s. Use these values to sketch the magnitude and phase response of the circuit.
(**Hint:** $4 + j \approx 4.1\angle 14°$, $\quad 1 + 4j \approx 4.1\angle 76°$)

Part III [2 points] Using what you know about frequency response, compute the steady state response $V_o^{SS}(t)$ of this circuit when $V_i(t) = 2\cos(1000t)$ using the same values of R and C as in Part II.

Solution:

Part I

First convert the circuit to the s-domain to get the following circuit (.5 point)



The compute $V_A(s)$ using the voltage-divider formula

$$V_A(s) = \frac{R}{R + 1/(sC)} V_i(s) = \frac{s}{s + 1/(RC)} V_i(s) \qquad \text{(1 point)}$$

and $V_B(s)$ using the formula for the non-inverting Op-Amp gain

$$V_B(s) = \frac{2R + R}{R} V_A(s) = \frac{3s}{s + 1/(RC)} V_i(s) \qquad \text{(1 point)}$$

In order to compute $V_o(s)$ we have to write KCL at V_o

$$\frac{V_o(s) - V_i(s)}{R} + \frac{V_o(s) - V_B(s)}{R} + \frac{V_o(s)}{R/2} = 0 \qquad \text{(1 point)}$$

That is

$$4V_o(s) = V_i(s) + V_B(s) = \left[1 + \frac{3s}{s + 1/(RC)}\right] V_i(s) = \frac{4s + 1/(RC)}{s + 1/(RC)} V_i(s)$$

from where

$$V_o(s) = \frac{s + 1/(4RC)}{s + 1/(RC)} V_i(s) \qquad \text{(.5 point)}$$

Part II

With $R = 10$ KΩ, $C = 100$ nF we have

$$\omega_c := \frac{1}{RC} = \frac{1}{10 \times 10^3 \times 10^2 \times 10^{-9}} = 10^3 \text{ rad/s}$$

so that

$$T(s) = \frac{s + \omega_c/4}{s + \omega_c} = \frac{s + 250}{s + 1000} \qquad \text{(1 point)}$$

At $s = j0$ and $s = j\infty$

$$T(j0) = \frac{250}{1000} = \frac{1}{4}, \qquad T(j\infty) = \frac{j\infty}{j\infty} = 1 \qquad \text{(1 point)}$$

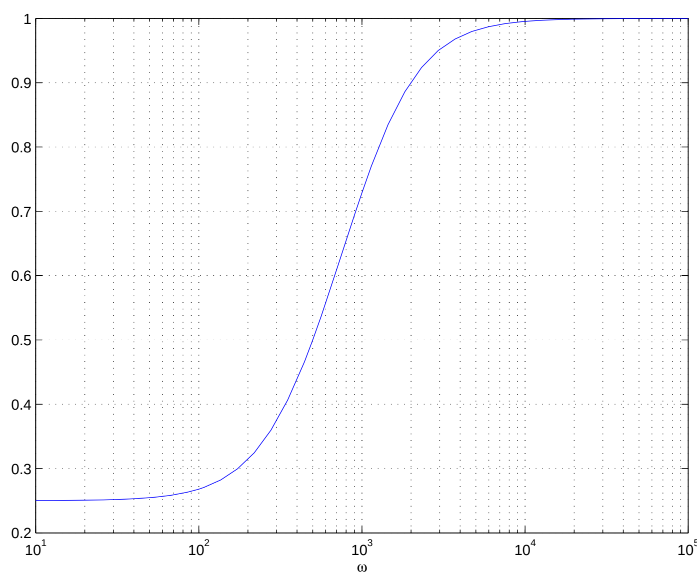
At $s = j250$ and $s = j1000$

$$T(j250) = \frac{250 + j250}{1000 + j250} = \frac{1 + j}{4 + j}, \qquad T(j1000) = \frac{250 + j1000}{1000 + j1000} = \frac{1 + 4j}{4(1 + j)}$$

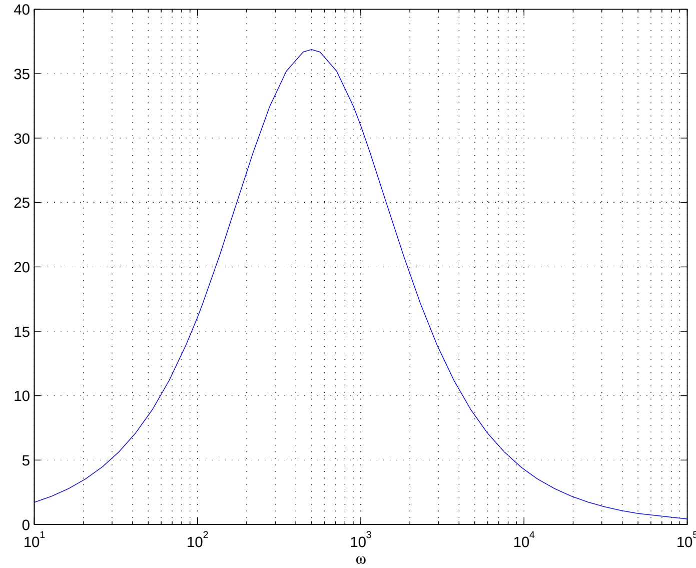
which converted to polar form is

$$T(j250) = \frac{1 + j}{4 + j} \approx \frac{\sqrt{2}}{4.1}\angle(45^\circ - 14^\circ) = 0.35\angle 31^\circ, \qquad \text{(1 point)}$$
$$T(j1000) = \frac{1 + 4j}{4(1 + j)} \approx \frac{4.1}{4\sqrt{2}}\angle(76^\circ - 45^\circ) = 0.72\angle 31^\circ$$

The magnitude response is (.5 point)



The phase response is (.5 point)



Part III

In order to compute the steady state response we use the frequency response of the circuit, that is

$$V_o^{SS}(t) = 2|T(j1000)|\cos(1000t + \angle T(j1000)) \qquad \text{(1 point)}$$

Borrowing from the previous part

$$V_o^{SS}(t) \approx 2 \times 0.72\ \cos(1000t + 31^\circ) = 1.44\ \cos(1000t + 31^\circ) \qquad \text{(1 point)}$$

5. **Op-Amp Circuit Analysis and Design**

When you take MAE 143 B you will study a very popular controller called a *proportional-integral-derivative controller* (PID-Controller). As the name suggests, a PID-Controller transforms a signal $y(t)$ into the control signal $u(t)$ through the formula

$$u(t) = b\,y(t) + c\int_0^t y(\tau)d\tau + a\,\frac{d}{dt}y(t)$$

where the parameters a, b and c are chosen by an engineer.

Part I [4 points] Assuming zero initial conditions, show that the transfer function from $Y(s)$ to $U(s)$ is given by

$$T(s) = \frac{U(s)}{Y(s)} = \frac{as^2 + bs + c}{s}$$

Show your work, indicating the simplifications implied by the assumption of zero initial conditions.

Part II [4 points] Now let $a = 0$ and $b < 1$. Design a circuit using one capacitor, one OpAmp and multiple resistors that realizes $T(s)$. If you don't know how to do that using a single OpAmp, try doing it with two OpAmps still using one capacitor for 3 points. Clearly indicate the relationship between your circuit components and b and c, making sure $b < 1$.

Part III [2 points] Find values for your components so that $a = 0$, $b = .5$ and $c = 0.05$.

Solution:

Part I

Apply Laplace transforms to both sides

$$\mathcal{L}\{u(t)\} = \mathcal{L}\left\{b\,y(t) + c\int_0^t y(\tau)d\tau + a\,\frac{d}{dt}y(t)\right\} \qquad \text{(1 point)}$$

to get

$$U(s) = b\,Y(s) + c\,\frac{1}{s}Y(s) + a\,[sY(s) - y(0)] \qquad \text{(1 point)}$$

Now because $y(0) = 0$ we have

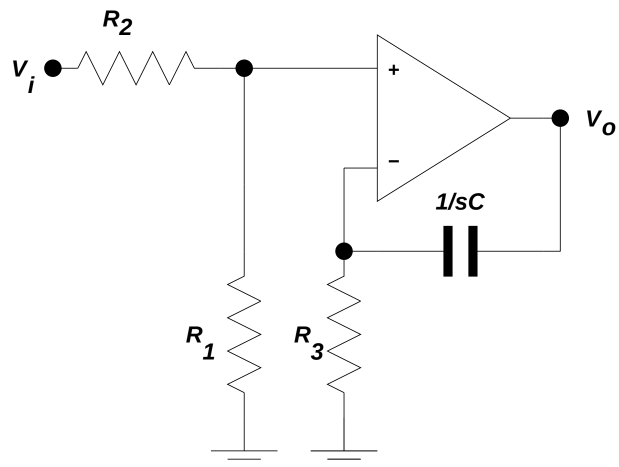
$$U(s) = \left[as + b + c\,\frac{1}{s}\right]Y(s) \qquad \text{(1 point)}$$

from where

$$\frac{U(s)}{Y(s)} = \frac{as^2 + bs + c}{s} \qquad \text{(1 point)}$$

Part II

Use the circuit



which is a voltage divider with R_1 and R_2, hence gain

$$\frac{R_1}{R_1 + R_2} \qquad \text{(1 point)}$$

followed by a non-inverting OpAmp block with $Z_3(s) = R_3$ and $Z_4(s) = 1/(sC)$ so that

$$\begin{aligned} T(s) &= \frac{R_1}{R_1 + R_2} \frac{Z_3(s) + Z_4(s)}{Z_3(s)} = \frac{R_1}{R_1 + R_2} \frac{R_3 + 1/(sC)}{R_3} \\ &= \frac{R_1}{R_1 + R_2} \frac{s + 1/(R_3 C)}{s} = \frac{bs + a}{s} \end{aligned} \qquad \text{(2 points)}$$

for

$$b := \frac{R_1}{R_1 + R_2} < 1, \qquad a := \frac{R_1}{(R_1 + R_2) R_3 C}. \qquad \text{(1 point)}$$

Part III

One solution is having $R_1 = R_2$ and $R_3 C = 10$ so that

$$a = \frac{R_1}{(R_1 + R_2) R_3 C} = \frac{1}{20} = 0.05, \qquad b = \frac{R_1}{R_1 + R_2} = 0.5. \qquad \text{(1 point)}$$

Now chose for example $R_1 = R_2 = 100\ \text{K}\Omega$ and $R_3 = 1\ \text{M}\Omega$, $C = 10\ \mu\text{F}$. (1 point)