MAE 140 – Linear Circuits – Fall 2008
Final

Instructions

1) This exam is open book. You may use whatever written materials you choose, including your class notes and textbook. You may use a hand calculator with no communication capabilities.

2) You have 170 minutes.

3) On the questions for which we have given the answers, please provide detailed derivations.
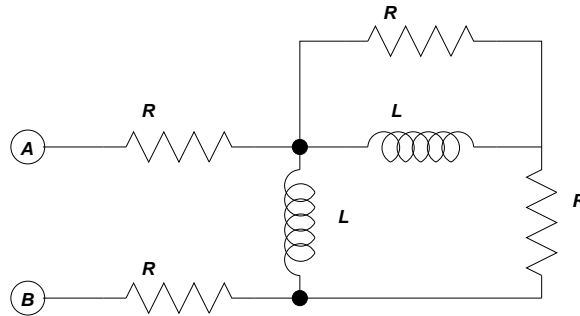
Question 1 — Equivalent Circuits

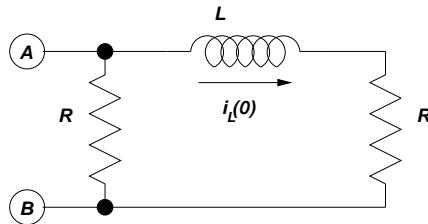Figure 1: Circuit for Question 1 (i)

Figure 2: Circuit for Question 1 (ii)

Part (i) [5 marks] Assuming zero initial conditions, find the impedance equivalent to the circuit in Figure 1 as seen from terminals A and B. The answer should be given as a ratio of two polynomials.

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Compute

$$Z(s) = R + R + [sL // (R + sL//R)]$$

$$= 2R + \left[sL // \left(R + \frac{sLR}{sL + R} \right) \right]$$

$$= 2R + \left[sL // \left(\frac{2sLR + R^2}{sL + R} \right) \right]$$

$$= 2R + \left[\frac{sL \left(\frac{2sLR + R^2}{sL+R} \right)}{sL + \left(\frac{2sLR + R^2}{sL+R} \right)} \right]$$

$$= 2R + \left(\frac{2s^2 L^2 R + sLR^2}{s^2 L^2 + 3sLR + R^2} \right)$$

$$= \frac{4s^2 L^2 R + 7sLR^2 + 2R^3}{s^2 L^2 + 3sLR + R^2}$$

[+5 marks; partial marks for partial work]

Part (ii) [5 marks] Assuming that the initial condition of the inductor is as indicated in the diagram, redraw the circuit shown in Figure 2 in the s-domain. Then use source transformations to find the s-domain Thevenin equivalent of this circuit as seen from terminals A and B.
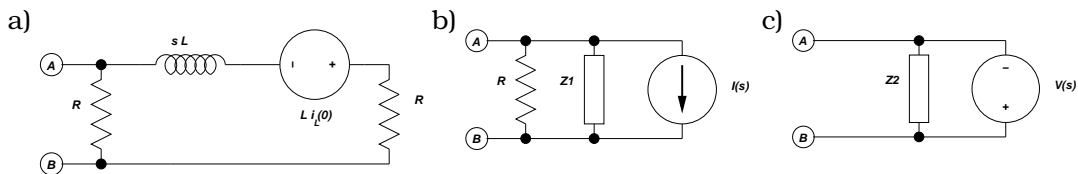
a)   b)   c)

Figure 3: Sequence of Transformations – Question 1 (ii)

First transform the circuit to the s-domain to include the voltage source $Li_L(0)$ in series with the impedance sL as in Figure 3a).

[+2 marks]

Associate the inductor and resistor into the impedance

$$Z_1(s) = sL + R$$

and convert the voltage source into a current source

$$I(s) = \frac{Li_L(0)}{Z_1(s)} = \frac{Li_L(0)}{sL + R}$$

in parallel with $Z_1(s)$ as in Figure 3b).

[+1 mark]

Associate R and $Z_1(s)$ into the impedance

$$Z_2(s) = Z_1(s)//R = \frac{R(sL + R)}{2R + sL}$$

[+1 mark]

Then convert the current source in parallel with $Z_2(s)$ into the voltage source

$$V(s) = I(s)Z_2(s) = \frac{Li_L(0)}{sL + R}\frac{R(sL + R)}{2R + sL} = \frac{RLi_L(0)}{2R + sL}$$

to obtain the Thevenin equivalent circuit shown in Figure 3c).

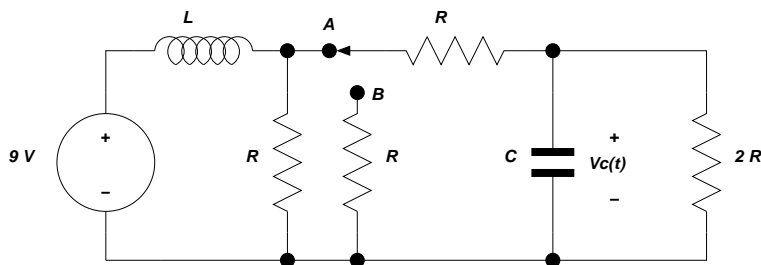[+1 mark]

Question 2 — Laplace Domain Circuit Analysis

Figure 4: RC circuit for Laplace analysis.

Part (i) [3 marks] Consider the circuit depicted in Figure 4. The voltage source is constant. The switch is kept in position A for a very long time. At $t = 0$ it is moved to position B. Show that the initial capacitor voltage is given by

$$v_C(0^-) = 6V.$$

[Show your work.]

With a constant input and the switch in position A for a very long time, the circuit reaches steady state. In steady state, the circuit is described by all time derivatives of signals being zero. In particular, $i_C(t) = \frac{dv_C(t)}{dt} = 0$ and $v_L(t) = \frac{di_L(t)}{dt} = 0$.

[+2 marks]

Therefore, the steady state current through the capacitor is zero. In this condition, the steady-state voltage $v_C(0^-)$ is given by the voltage divider

$$v_C(0^-) = \frac{2R}{2R + R}9 = 6V$$

[+1 mark]

Part (ii) [2 marks] Use this initial condition to transform the right side of the circuit (the one that contains the capacitor) into the s-domain for $t \geq 0$. Use an equivalent model for the capacitor in which the initial condition appears as a voltage source.
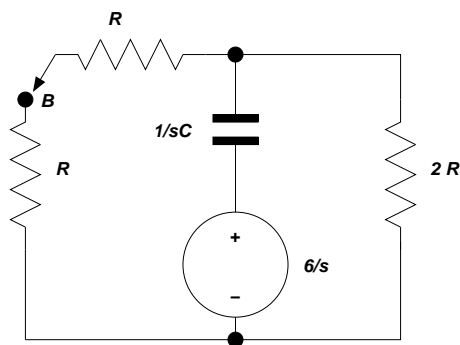[Show your work.]

Figure 5: s-domain circuit.

With the initial voltage from Part (i), redraw the circuit for time $t \geq 0$ as in Figure 5. Note that the initial capacitor voltage $v_C(0-)$ appears as a voltage source $6/s$ in series with the impedance $1/(sC)$.

[+2 marks]

Part (iii) [3 marks] Use s-domain circuit analysis and inverse Laplace transforms to show that the capacitor voltage satisfies,

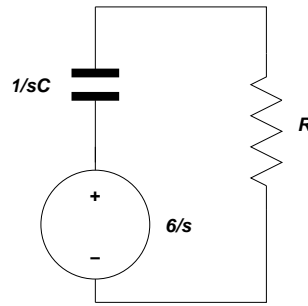$$v_C(t) = 6\,e^{-t/(RC)}\,u(t).$$

[Show your work.]

Figure 6: Simplified s-domain circuit.

First combine the two resistors in the left into a $2R$ resistor then combine the two $2R$ resistors in parallel to obtain the resistor R and the circuit in Figure 6.

[+1 mark]

Write KVL to obtain

$$I(s) = \frac{6}{s}\frac{1}{R+1/sC} = \frac{6}{s}\frac{s1/R}{s+1/RC} = \frac{6/R}{s+1/RC}$$

then

$$\begin{aligned}V_C(s) &= \frac{6}{s} - \frac{1}{sC}I(s) \\ &= \frac{6}{s} - \frac{6/(RC)}{s[s+1/(RC)]} = \frac{6[s+1/(RC)]-6/(RC)}{s[s+1/(RC)]} = \frac{6s}{s[s+1/(RC)]} = \frac{6}{s+1/(RC)}\end{aligned}$$

[+1 mark]

Taking inverse Laplace transforms,

$$v_C(t) = 6e^{-t/(RC)}u(t).$$

[+1 mark]

Part (iv) [2 marks + 2 bonus marks] Using what you know about s-domain analysis, compute the voltage and current across the inductor (in the left side of the circuit) when the switch is moved from A to B. Is this a safe circuit?

For the circuit on the left $v_L(0-) = 0$ and

$$i_L(0-) = \frac{9}{R} + \frac{9}{3R} = \frac{12}{R}$$

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[+1 mark]

Furthermore

$$I_L(s) = \frac{9/s + Li_L(0)}{R + sL} = \frac{9/s + 12L/R}{R + sL} = \frac{9/L + 12/Rs}{s(s + R/L)}$$

[+1 mark]

By partial fractions expandion

$$I_L(s) = \frac{9/L + 12/Rs}{s(s + R/L)} = \frac{\alpha}{s} + \frac{\beta}{s + R/L}$$

where

$$\alpha = \frac{9/L}{R/L} = \frac{9}{R}, \qquad \beta = \frac{9/L + 12/R(-R/L)}{(-R/L)} = \frac{12 - 9}{R} = \frac{3}{R}$$

hence

$$i_L(t) = \frac{1}{R}\left[9 + 3e^{-t(R/L)}\right]u(t)$$

and

$$v_L(t) = L\frac{d}{dt}i_L(t) = -\frac{3}{L}e^{-t(R/L)}u(t)$$

[+1 bonus mark]

Note that this type analysis hides the fact that there is a discontinuity in the current of the inductor. Therefore, at the point of discontinuity, the voltage

$$v_L(t) = L\frac{d}{dt}i_L(t)$$

will be likely very large, even if everything else seems OK after the discontinuity (from the above analysis). With high currents this may be a threat, or may damage the inductor or other parts of the circuit.

[+1 bonus mark]
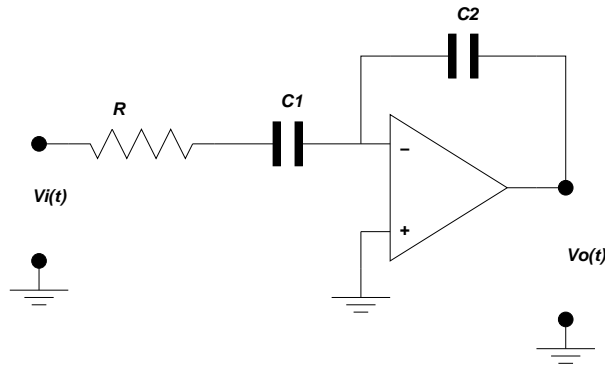
Question 3 — Active Filter Analysis and Design

Figure 7: Filter for analysis

Part (i) [2 marks] Assuming zero initial conditions, transform the circuit in Figure 7 into the s-domain and compute the transfer function from $V_i(s)$ to $V_o(s)$.
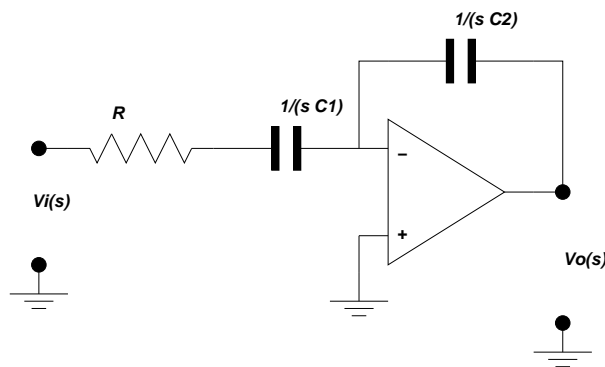

Figure 8: Filter in the s-domain

First transform the circuit into the s-domain as in Figure 8.

[+1 mark]

Then

$$T(s) = V_o(s)/V_i(s) = -\frac{1/(sC_2)}{R + 1/(sC_1)} = -\frac{1/(RC_2)}{s + 1/(RC_1)}$$

[+1 mark]

Part (ii) [3 marks] Showing your reasoning, determine the nature of this filter's frequency response. Further, determine the gain of the filter and its cut-off frequency.

This is a first order filter so that the filter's characteristic can be obtained evaluating $T(j\omega)$ as $\omega \to 0$ and $\omega \to \infty$. In this case

$$\lim_{\omega \to 0} T(j\omega) = -\frac{1/(RC_2)}{1/(RC_1)} = -\frac{C_1}{C_2}, \qquad\qquad \lim_{\omega \to \infty} T(j\omega) = 0.$$

Therefore it is a low-pass filter.

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[+1 mark]

The cut-off frequency coincides with the absolute value of the pole

$$\omega_c = 1/(RC_1).$$

[+1 mark]

The gain is the peak value of the absolute value of the frequency response which is at $\omega = 0$

$$K = |\lim_{\omega \to 0} T(j\omega)| = \frac{C_1}{C_2}$$

[+1 mark]
[You also get marks if you associate the transfer function with the expressions in the book and invoque the book's results, as long as they are correct. If you have $-C_1/C_2$ as gain you also get marks.]

Part (iii) [2 marks] If $C_1 = C_2 = 100$nF, find the value of R so that the cutoff frequency is 5 KHz.

We need $\omega_c = 2\pi \times 5 \times 10^3 = \pi \times 10^4$.

[+1 mark]

With $C_1 = 100nF = 10^{-7}F$ we have

$$\omega_c = \pi \times 10^4 = \frac{1}{R \times 10^{-7}} \quad \Longrightarrow \quad R = \frac{1}{\pi \times 10^{-3}} = \frac{1}{\pi} K\,\Omega$$

[+1 mark]

Part (iv) [3 marks] Using $C_1 = C_2 = 100$nF and the value of R you computed in the previous item, design a second stage op-amp circuit that if connected to $V_o(t)$ will make the overal filter gain (from $V_i(s)$ to the output of the second stage) be equal to 1.

[Attention: the gain equal to -1 is not correct!]

For $C_1 = C_2$ gain is -1. Hence an inverter with unit gain will make the overal gain equal to one. The circuit is as in Figure 9.
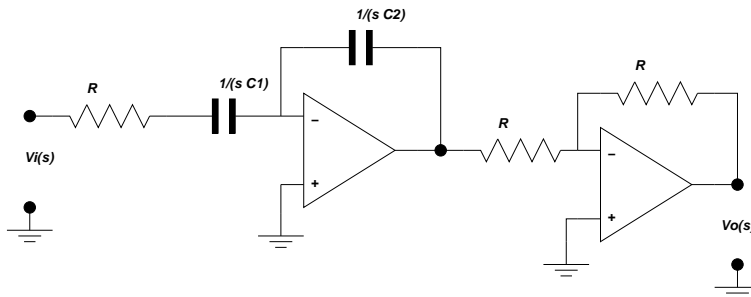

Figure 9: Filter in the s-domain

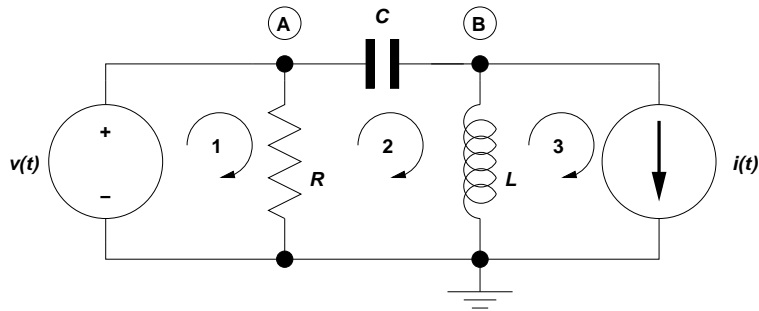*[+3 marks]*

Question 4 — Nodal and Mesh Analysis

Figure 10: Nodal and Mesh Analysis Circuit

Part (i) [5 marks] Formulate node-voltage equations in the s-domain for the circuit in Figure 10. Use the reference node and other labels as shown in the figure. Do not assume zero initial conditions! Transform initial conditions on the capacitor and on the inductor into current sources.

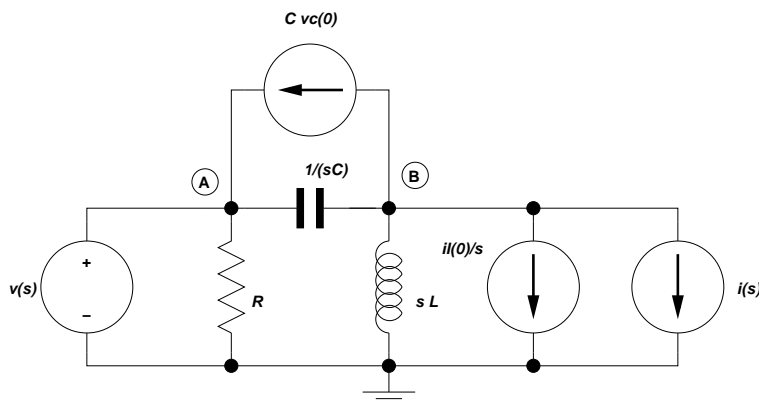*First transform the circuit into the s-domain as in Figure 11.*

Figure 11: Circuit in the s-domain

[+1 mark for correct circuit]

For the above circuit we have

$$v_C(0) = V_A(0) - V_B(0),$$

and $i_L(0)$ pointing down.

[+1 mark for correct initial conditions]

[As I did not indicate the polarity and the direction in the circuit you have to pick one. You get marks in this item (i) even if you do not indicate them explicitly.]

Note that node A is connected to a voltage source so that

$$V_A(s) = v(s)$$

[+ 1 mark]

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Furthermore we need only to write the node-voltage equation at node B by inspection

$$\begin{bmatrix} -sC & sC + \frac{1}{sL} \end{bmatrix} \begin{pmatrix} V_A(s) \\ V_B(s) \end{pmatrix} = -Cv_C(0) - \frac{i_L(0)}{s} - i(s)$$

or using KCL.

The final equation to be solved for $V_B(s)$ is

$$\left(sC + \frac{1}{sL} \right) V_B(s) = sCv(s) - Cv_C(0) - \frac{i_L(0)}{s} - i(s)$$

[+2 marks for correct equations]

Part (ii) [5 marks] Formulate mesh-current equations in the s-domain for the circuit in Figure 10. Use the currents shown in the figure. Do not assume zero initial conditions! Transform initial conditions on the capacitor and on the inductor into voltage sources.

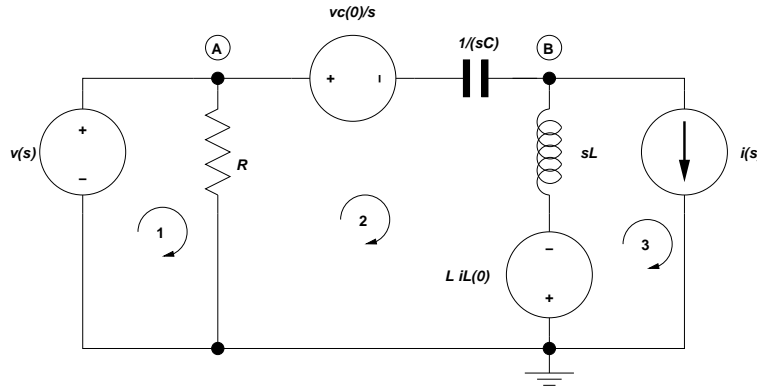*First transform the circuit into the s-domain as in Figure 12.*

Figure 12: Circuit in the s-domain

[+1 mark for correct circuit]
[+1 mark for correct initial conditions]
[You lose 1 mark if the initial conditions are not compatible with the one you chose in item (i).]
Note that in this circuit

$$I_3(s) = i(s)$$

[+1 mark]
So we have to write the mesh-current equations by inspection only for meshes 1 and 2

$$\begin{bmatrix} R & -R & 0 \\ -R & R + \frac{1}{sC} + sL & -sL \end{bmatrix} \begin{pmatrix} I_1(s) \\ I_2(s) \\ I_3(s) \end{pmatrix} = \begin{pmatrix} V(s) \\ Li_L(0) - \frac{v_C(0)}{s} \end{pmatrix}$$

or one-by-one using KVL.
The final equations to be solved for $I_1(s)$ and $I_2(s)$ are

$$\begin{bmatrix} R & -R \\ -R & R + \frac{1}{sC} + sL \end{bmatrix} \begin{pmatrix} I_1(s) \\ I_2(s) \end{pmatrix} = \begin{pmatrix} V(s) \\ Li_L(0) - \frac{v_C(0)}{s} + sLi(s) \end{pmatrix}$$

[+2 marks for correct equations]

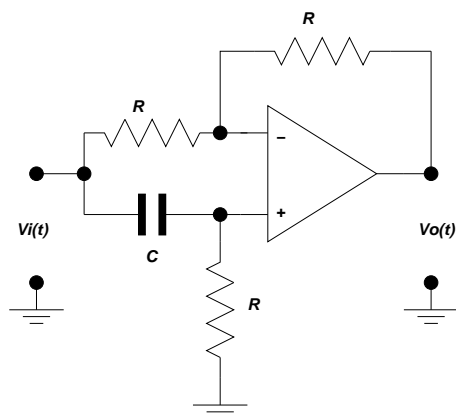## Question 5 — Op-Amp Analysis and Application

Figure 13: Op-Amp Circuit

Part (i) [5 marks] Using the fundamental op-amp relationships, find the transfer function $T(s)$ from $V_i(s)$ to $V_o(s)$ in Figure 13. Assume zero initial conditions.

Because $I_+ = I_- = 0$ we have that

$$V_-(s) = V_+(s) = \frac{R}{R + 1/(sC)} V_i(s) = \frac{s}{s + 1/(RC)} V_i(s)$$

where the last comes from a voltage divider.

[+ 2 marks]

Then the current $I(s)$ in the top branch becomes

$$I(s) = [V_i(s) - V_-(s)]/R$$

$$= (1/R)V_i(s) - \frac{s(1/R)}{s + 1/(RC)} V_i(s)$$

$$= \frac{1}{R} \frac{s + 1/(RC) - s}{s + 1/(RC)} V_i(s) = \frac{1}{R} \frac{1/(RC)}{s + 1/(RC)} V_i(s)$$

[+ 1 mark]

Finnaly $I(s) = [V_-(s) - V_o(s)]/R$ so that we have

$$V_o(s) = V_-(s) - RI(s)$$

$$= \frac{s}{s + 1/(RC)} V_i(s) - \frac{1/(RC)}{s + 1/(RC)} V_i(s) = \frac{s - 1/(RC)}{s + 1/(RC)} V_i(s)$$

[+2 marks]

Part (ii) [5 marks] Compute the magnitude and phase of the transfer function $T(j\omega)$, that is $|T(j\omega)|$ and $\angle T(j\omega)$. Describe in words what is the function of this circuit.

[Hint: Use the frequency response of the circuit to see what happens to an input of the form $V_i(t) = A\cos(\omega t)$.]

From Part (i),

$$T(s) = \frac{s - 1/(RC)}{s + 1/(RC)}$$

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For $s = j\omega$

$$T(j\omega) = \frac{j\omega - 1/(RC)}{j\omega + 1/(RC)}$$

Then

$$|T(j\omega)| = \frac{|j\omega - 1/(RC)|}{|j\omega + 1/(RC)|} = 1$$

[+ 2 marks]

For the phase

$$\angle T(j\omega) = \angle[j\omega - 1/(RC)] - \angle[j\omega + 1/(RC)] = -2\tan^{-1}(\omega RC)$$

[+ 2 marks]

The circuit is an all pass filter. *It does not change the magnitude response to sinusoidal inputs, just the phase. Namely, if $V_i(t) = A\cos(\omega t)$ then the output of the filter is*

$$V_o(t) = A|T(j\omega)|\cos[\omega t + \angle T(j\omega)] = A\cos[\omega t - 2\tan^{-1}(\omega RC)]$$

[+ 1 mark]

The phase varies from 0 to $-180°$.

Part (iii) [2 bonus marks] Can you describe a possible application for this circuit?

This circuit is used to compensate the phase without affecting the magnitude of the frequency response. For example, it could be used to compensate the phase of a signal that has been distorted by filtering.

[+ 2 marks]