MAE 140 – Linear Circuits – Fall 2007
Final

Instructions

1) This exam is open book. You may use whatever written materials you choose, including your class notes and textbook. You may use a hand calculator with no communication capabilities.
2) You have 170 minutes.
3) On the questions for which we have given the answers, please provide detailed derivations.

Question 1 — Equivalent Circuits

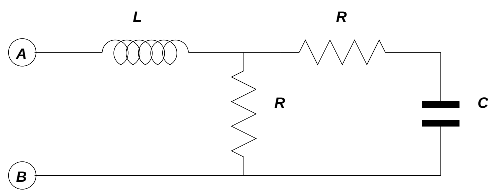

Figure 1: Circuit for Question 1 (i)

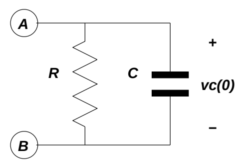

Figure 2: Circuit for Question 1 (ii)

Part (i) [5 marks] Assuming zero initial conditions, find the impedance equivalent to the circuit in Figure 1 as seen from terminals A and B. The answer should be given as a ratio of two polynomials.

Compute

$$\begin{aligned} Z(s) &= sL + \left[R // \left(R + \frac{1}{sC}\right)\right] \\ &= sL + \frac{R\left(R + \frac{1}{sC}\right)}{2R + \frac{1}{sC}} \\ &= sL + \frac{R\,(sRC+1)}{2sRC+1} \\ &= \frac{sL(2sRC+1) + R\,(sRC+1)}{2sRC+1} \\ &= \frac{2s^2RLC + s(L+R^2C) + R}{2sRC+1} \end{aligned}$$

Part (ii) [5 marks] Assuming that the initial condition of the capacitor is as indicated in the diagram, redraw the circuit shown in Figure 2 in s-domain. Then use source transformations to find the s-domain Thévenin equivalent to the circuit as seen from terminals A and B.

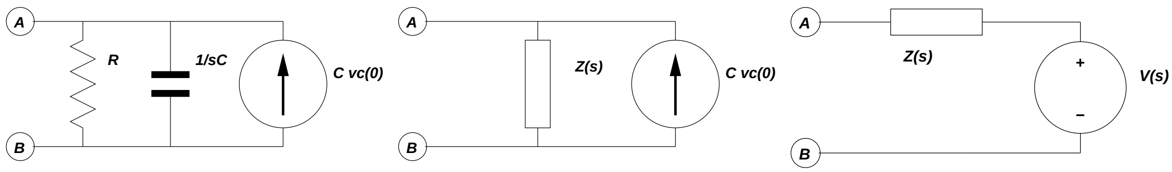

Figure 3: Sequence of Transformations – Question 1 (ii)

First transform the circuit to the s-domain to include the current source $I(s) = Cv_c(0)$ in parallel with $1/(s\,C)$. Associate the capacitor and resistor into the impedance

$$Z(s) = \frac{\frac{R}{s\,C}}{R + \frac{1}{s\,C}} = \frac{R}{1 + sRC}$$

The Thévenin equivalent circuit is obtained after transforming the current source into a voltage source

$$V(s) = Z(s)I(s) = \frac{RCv_c(0)}{1 + sRC}.$$

This sequence of transformations is shown in Figure 3.

Question 2 — Laplace domain circuit analysis

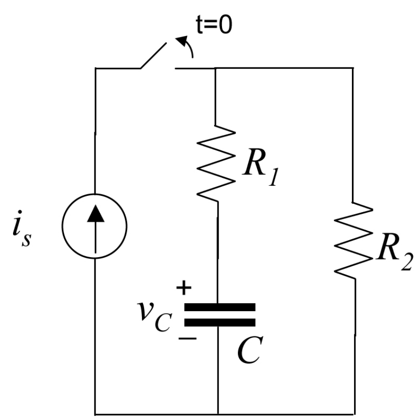

Figure 4: RC circuit for Laplace analysis.

Part (i) [3 marks] Consider the circuit depicted in Figure 4. The current source i_s is a constant current supply, which is kept in place for a very long time until the switch is opened at time $t = 0$. Show that the initial capacitor voltage is given by

$$v_C(0^-) = i_s R_2.$$

[Show your working.]

With a constant input and the switch closed for a very long time, the circuit reaches steady state. In steady state, the circuit is described by all time derivatives of signals being zero. In particular, $\frac{dv_C(t)}{dt} = 0$. Therefore, the steady state current through the capacitor is zero, as is the current through the resistor R_1. So, all of the current i_s flows through resistor R_2, yielding voltage across that resistor of $R_2 i_s$ volts. Since there is no current through R_1, all of this voltage drop appears across capacitor C. Hence, $v_C(0^-) = i_s R_2$.

Part (ii) [2 mark] Use this initial condition to transform the circuit into the s-domain for time $t \geq 0$. [The symbolic formulæ are shown in the text on page 449.] [Show your working.]

With the initial voltage from Part (i), redraw the circuit for time $t \geq 0$ as below. Note that

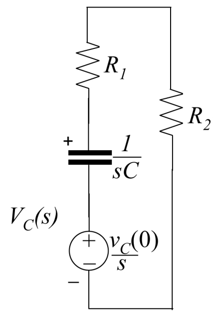

Figure 5: s domain RC circuit.

the capacitor voltage $V_c(s)$ includes both the equivalent impedance $\frac{1}{sC}$ and the series initial condition source $\frac{v_C(0^-)}{s}$.

Part (iii) [5 marks] Use s-domain circuit analysis and inverse Laplace transforms to show that the capacitor voltage satisfies,

$$v_C(t) = i_s R_2 \exp\left(\frac{-t}{(R_1 + R_2)C}\right) u(t).$$

[Show your working.]

The capacitor voltage $V_c(s)$ is the same as the voltage across the $R_1 + R_2$ resistor pair. Use the voltage divider formula to yield

$$\begin{aligned} V_c(s) &= \frac{R_1 + R_2}{R_1 + R_2 + \frac{1}{sC}} \frac{v_C(0^-)}{s}, \\ &= \frac{R_1 + R_2}{(R_1 + R_2)s + \frac{1}{C}} i_s R_2, \\ &= \frac{i_s R_2}{s + \frac{1}{(R_1+R_2)C}}. \end{aligned}$$

Hence, taking inverse Laplace transforms,

$$v_C(t) = i_s R_2 \exp\left(\frac{-t}{(R_1 + R_2)C}\right) u(t).$$

Question 3 — Active Filter Analysis and Design

Part (i) [3 marks] Show that the the transfer functions of the op-amp circuits in Figure 6 are given by

$$T_{RC}(s) = \frac{-1}{R_2} \times \frac{\frac{1}{C}}{s + \frac{1}{R_1 C}}, \qquad T_{RL}(s) = \frac{-1}{R_4} \times \frac{sR_3}{s + \frac{R_3}{L}}.$$

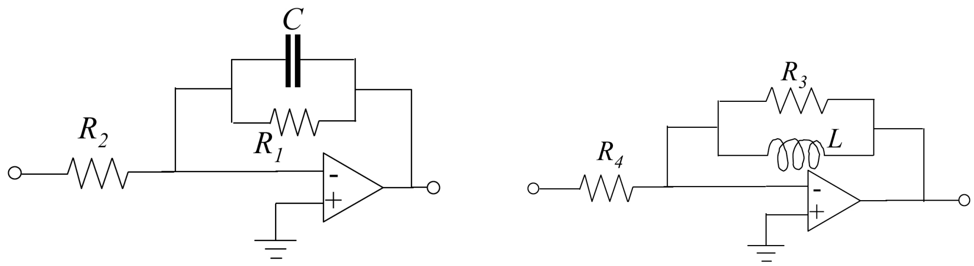

Figure 6: (a) RC parallel op-amp circuit. (b) RL parallel op-amp circuit.

The RC parallel impedance is given by

$$Z_{RC}(s) = \frac{R_1 \times \frac{1}{sC}}{R_1 + \frac{1}{sC}} = \frac{\frac{1}{C}}{s + \frac{1}{R_1 C}}.$$

The op-amp circuit is in the inverting amplifier configuration and so its transfer function is given by the ratio of impedances, with $Z_{RC}(s)$ in the numerator position. Hence,

$$T_{RC}(s) = \frac{-1}{R_2} \times \frac{\frac{1}{C}}{s + \frac{1}{R_1 C}}.$$

For the RL parallel impedance,

$$Z_{RL}(s) = \frac{sLR_3}{sL + R_3} = \frac{sR_3}{s + \frac{R_3}{L}}.$$

We also have an inverting op-amp configuration. So,

$$T_{RL}(s) = \frac{-1}{R_4} \times \frac{sR_3}{s + \frac{R_3}{L}}.$$

Part (ii) [5 marks] Showing your reasoning, determine the nature of these two filters' frequency responses. Further, determine the gain of the filters and their cut-off frequencies.

If $C = 100$nF, find R values so that the RC filter has cutoff frequency 5KHz and gain 5.

If $L = 10$mH, find R values so that the RL filter has cutoff frequency 1KHz and gain 5.

To generate the frequency response, replace the Laplace variable s by $j\omega$, where ω is the radian frequency. Thus,

$$T_{RC}(j\omega) = \frac{-1}{R_2} \times \frac{\frac{1}{C}}{j\omega + \frac{1}{R_1 C}}.$$

Now compute the magnitude of this frequency response

$$\begin{aligned} |T_{RC}(j\omega)| &= \left|\frac{-1}{R_2}\right| \times \frac{|\frac{1}{C}|}{|j\omega + \frac{1}{R_1 C}|}, \\ &= \frac{\frac{1}{R_2 C}}{\sqrt{\omega^2 + \frac{1}{(R_1 C)^2}}}. \end{aligned}$$

Clearly the denominator strictly increases with frequency ω, while the numerator stays fixed. Therefore the filter has a frequency response gain which decreases with frequency. We have at low frequencies,

$$\lim_{\omega \to 0} |T_{RC}(j\omega)| = \frac{R_1}{R_2}.$$

This is the resistive op-amp gain without the capacitor, because at low frequencies the capacitor is like an open circuit. For very high frequencies, we have

$$\lim_{\omega\to\infty} |T_{RC}(j\omega)| = 0.$$

Therefore the filter is a low-pass filter. The cut-off frequency is given by pole value, which is where the gain is $\frac{R_1}{R_2\sqrt{2}}$*.*

$$\omega_{\text{cutoff}} = \frac{1}{R_1 C}.$$

For the low-pass filter, the gain is given by $|T(0)| = \frac{R_1}{R_2}$*.*
Similarly, for the RL *filter*

$$\begin{aligned} |T_{RL}(j\omega)| &= \left|\frac{-1}{R_4}\right| \frac{|j\omega R_3|}{|j\omega + \frac{R_3}{L}|}, \\ &= \frac{R_3}{R_4}\frac{\omega}{\sqrt{\omega^2 + (R_3/L)^2}}. \end{aligned}$$

We have

$$\lim_{\omega\to 0} |T_{RL}(j\omega)| = 0, \quad \lim_{\omega\to\infty} |T_{RL}(j\omega)| = \frac{R_3}{R_4},$$

and so is a high-pass filter. The cutoff frequency is

$$\omega_{\text{cutoff}} = \frac{R_3}{L}.$$

For the high-pass filter, the gain is given by $\lim_{\omega\to\infty} |T(j\omega)| = \frac{R_3}{R_4}$*.*
To solve for the cut-off and gain values

$$\begin{aligned} \frac{1}{R_1 C} &= 5000 \times 2\pi, \\ \textit{or } R_1 &= 318\Omega, \\ \frac{R_3}{L} &= 1000 \times 2\pi, \\ \textit{or } R_3 &= 63\Omega, \\ \frac{R_1}{R_2} &= 5, \\ \textit{or } R_2 &= 64\Omega, \\ \frac{R_3}{R_4} &= 5, \\ \textit{or } R_4 &= 16\Omega. \end{aligned}$$

Part (iii) [2 marks] You are required to build a bandpass filter with;

- lower cut-off frequency of 1KHz,
- upper cut-off frequency of 5KHz, and
- passband gain of 25.

Explain whether the two filters above can be used to achieve this design specification. If so, then show how. If not, then suggest how the above calculations might be changed to yield such a design.

To design a bandpass filter as specified, we need to reject high frequencies above 5KHz — we will use the low-pass filter to do this — and to reject low frequencies below 1KHz, for *which we will use the high-pass filter. Therefore, we have already designed these filters. The bandpass filter is the cascade of the high-pass filter and the low-pass filter. The order is unimportant.*

The passband gain is then the product of the gains of the low-pass and high-pass filters.

$$G = \frac{R_1}{R_2} \times \frac{R_3}{R_4}.$$

This is achieved by this design.

Question 4 — Op-Amp Analysis and Application

The figure below shows a circuit known as a *gyrator* or *positive impedance inverter*. It is primarily used in active circuit design to implement inductors, which are difficult to manufacture to specification and within a small volume.

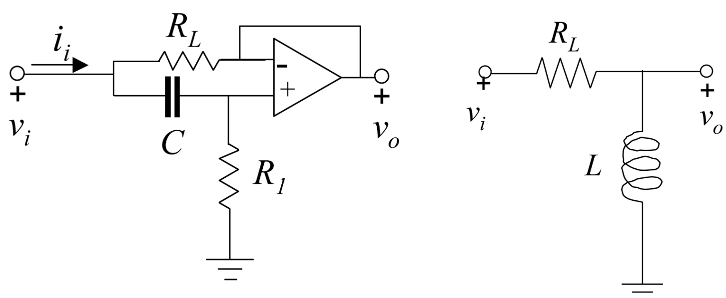

Figure 7: (a) Gyrator circuit. (b) RL-circuit.

Part (i) [5 marks] Using the fundamental op-amp relationships show that the voltage transfer function in Figure 7(a) is given by.

$$T_v(s) = \frac{V_o(s)}{V_i(s)} = \frac{s}{s + \frac{1}{R_1 C}}.$$

Because the current into the p*-terminal of the op-amp is zero, the* R_1, C *combination is a voltage divider on the input voltage* v_i*. Therefore,*

$$\begin{aligned} V_p(s) &= \frac{R_1}{R_1 + \frac{1}{sC}} V_i(s), \\ &= \frac{s}{s + \frac{1}{R_1 C}}. \end{aligned}$$

Likewise, using the op-amp relations $v_n = v_p$ *and the property that the output terminal is directly connected to the* n*-terminal.*

$$V_o(s) = V_n(s) = V_p(s) = \frac{s}{s + \frac{1}{R_1 C}} V_i(s).$$

Part (ii) [2 marks] Perform the same calculation for the RL-circuit shown in Figure 7(b) to show that its transfer function is

$$T_v(s) = \frac{sL}{sL + R_L}.$$

Thus, the gyrator circuit effectively implements a circuit involving an inductor by using a capacitor and an op-amp. Determine the equivalent inductor value L in the RL-circuit in terms of the element values, R_1, R_L, C, in the gyrator circuit

For the RL-circuit we also have a voltage divider. Thus, directly we have

$$T_v(s) = \frac{sL}{sL + R_L} = \frac{s}{s + \frac{R_L}{L}}.$$

This is the same as for Part (i) with

$$R_1 C = \frac{L}{R_L}, \quad \text{or } L = R_1 C R_L.$$

Part (iii) [3 marks] Show that the equivalent impedance of the gyrator circuit seen from the input, v_i, is given by

$$Z_{eq}(s) = \frac{sR_1 C R_L + R_L}{sR_L C + 1},$$

while, for the RL-circuit it is given by

$$Z_{eq}(s) = sL + R_L.$$

Computing the current i_i flowing into the gyrator circuit in response to the applied voltage v_i, we have

$$\begin{aligned}
I_i(s) &= \frac{1}{R_L}(V_i(s) - V_o(s)) + cS(V_i(s) - V_o(s)), \\
&= \left(\frac{1}{R_L} + sC\right)\left(V_i(s) - \frac{s}{s + \frac{1}{R_1 C}} V_i(s)\right), \\
&= \left(\frac{1}{R_L} + sC\right)\left(1 - \frac{s}{s + \frac{1}{R_1 C}}\right) V_i(s), \\
&= \left(\frac{1}{R_L} + sC\right) \frac{\frac{1}{R_1 C}}{s + \frac{1}{R_1 C}} V_i(s).
\end{aligned}$$

Inverting the relationship to compute the impedance yields

$$Z_{eq}(s) = \frac{V_i(s)}{I_i(s)} = \frac{sR_1 C R_L + R_L}{sR_L C + 1}.$$

The RL-circuit result is immediate.

Part (iv) [3 mark, Bonus] Describe the limitations in using this substitute circuit to realize an inductor in an application.

The main limitations to using this circuit — and indeed it is used very widely — are as follows.

(a) *The circuit only simulates an RL-circuit with the end of the inductor grounded. In terms of the possible filter applications, it is only the differential amplifier (subtractor) form which makes sense. This is a high-pass filter if the grounded element is an inductor.*

(b) *Because the impedance of the gyrator circuit does not match that of an RL-circuit, one cannot use the gyrator as a 'drop-in' replacement. If one needs an inductor as an impedance element, then this circuit does not achieve this. If the RL impedance is important to the operation of the input side, then different approaches need to be taken. One might typically protect the input side of the this circuit with a voltage follower.*

(c) *Because the inductor is implemented as an op-amp circuit, all the limitations of op-amps apply. Notably the limit on the output voltage to the supply values to the op-amp is important. Normally in operation, a physical inductor is able to sustain very high voltages due to rapidly changing currents – so-called flywheel effects. The gyrator cannot do this.*

Question 5 — Power Factor Compensation

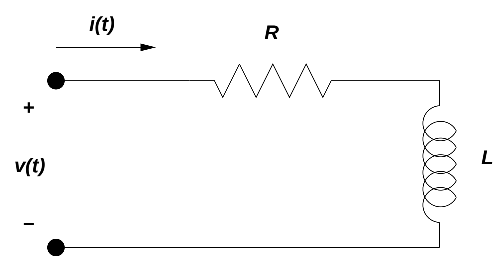

Figure 8: Motor equivalent RL circuit

Part (i) [3 marks] A large single phase alternating current (AC) induction electric motor can be represented by the RL circuit in Figure 8. Assuming zero initial conditions, find the equivalent impedance of the motor, $Z(s)$, and the equivalent admitance, $Y(s) = Z(s)^{-1}$. Compute $Y(j\omega)$, its real and imaginary parts. This is the admittance of the motor at the AC supply frequency of ω radians per second.

Compute the series association of R and L, that is $Z(s) = R + s\,L$, $Y(s) = 1/(R + sL)$. Then

$$\begin{aligned} Y(j\omega) &= \frac{1}{R + j\omega L} \\ &= \frac{R}{R^2 + \omega^2 L^2} + j\frac{-\omega L}{R^2 + \omega^2 L^2} \end{aligned}$$

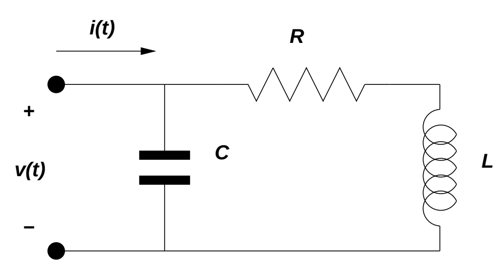

Figure 9: Motor equivalent RL circuit with capacitor C in parallel

Part (ii) [3 marks] Now consider the circuit in Figure 9, in which a capacitor is added in parallel to the motor. Assuming zero initial conditions, find the new equivalent admitance $Y_C(s)$. Compute $Y_C(j\omega)$, its real and imaginary parts. Use this to show that C may be chosen to make the imaginary part of the admittance zero at the supply frequency ω.

Compute

$$Y_C(s) = \frac{1}{R + s\,L} + s\,C$$

and then

$$\begin{aligned} Y_C(j\omega) &= \frac{1}{R + j\omega\,L} + j\omega\,C, \\ &= \frac{R - j\omega L}{R^2 + \omega^2\,L^2} + j\omega\,C, \\ &= \frac{R}{R^2 + \omega^2\,L^2} + j\omega\left(C - \frac{L}{R^2 + \omega^2\,L^2}\right). \end{aligned}$$

Choose $C = \frac{L}{R^2+\omega^2\,L^2}$ *to have zero imaginary part.*

Part (iii) [4 marks] If the supply voltage to the motor is $v(t) = V\cos(\omega t)$, write the corresponding sinusoidal steady state current $i(t)$ drawn by the machine with admittance $Y(s)$. Do this in the time domain as a $\cos$ function.

Compute the power $p(t) = v(t)i(t)$ and use the relationship $\cos(A)\cos(B) = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$, to derive an expression for the power absorbed by the motor as a function of time. Show that the average value of this power contains a term in $\mathrm{Re}(Y(j\omega)) = |Y(j\omega)|\cos[\angle(Y(j\omega))]$.

If a capacitor is chosen as in the previous item, does the introduction of this capacitor in parallel with the motor change the average power absorbed by the motor? Why?

Because $I(s) = Y(s)V(s)$, *by the frequency response formula,*

$$i_{ss}(t) = V|Y(j\omega)|\cos(\omega t + \angle(Y(j\omega))).$$

The power is given by the product above, whence

$$\begin{aligned} p(t) &= V^2|Y(j\omega)|\cos(\omega t)\cos(\omega t + \angle(Y(j\omega))), \\ &= \frac{V^2}{2}|Y(j\omega)|\left[\cos(2\omega t + \angle(Y(j\omega))) + \cos(\angle(Y(j\omega)))\right] \end{aligned}$$

The average of the first term is zero, since it is a cos function. The average of the second term is

$$< p > = \frac{V^2}{2}|Y(j\omega)|\cos(\angle(Y(j\omega))) = \frac{V^2}{2}\,\mathrm{Re}(Y(j\omega)).$$

Because the introduction of the capacitor does not change the real part of $Y(j\omega)$, *i.e.* $\mathrm{Re}(Y(j\omega)) = \mathrm{Re}(Y_C(j\omega))$, *it will not affect the average power absorbed by the motor.*

Part (iv) [3 mark, Bonus] Your power supplier (SDG&E and the like) will likely *demand* that you insert a capacitor C as above. They will enforce this by measuring $\cos[\angle(Y(j\omega))]$, aka *power factor*, and gently asking you to keep this number as close to one as possible, like you did by choosing C above. Why do you think these nice people would do that?

As shown in the previous question, the capacitor will not affect the average power absorbed by the motor, that is

$$< p > = \frac{V^2}{2}\,\mathrm{Re}(Y(j\omega)) = \frac{V^2}{2}\,\mathrm{Re}(Y_C(j\omega)).$$

but it will affect the magnitude of the current

$$i_{ss}(t) = V|Y(j\omega)|\cos(\omega t + \angle(Y(j\omega))).$$

Note that the power lost in the transmission wires $p_{\text{loss}}(t)$, say with resistance R_{loss}, is

$$\begin{aligned} p_{\text{loss}}(t) &= R_{\text{loss}} i_{ss}^2(t) \\ &= \frac{1}{2} R_{\text{loss}} V^2 |Y(j\omega)|^2 \left[\cos(2\omega t + \angle Y(j\omega)) + 1\right] \end{aligned}$$

which averages to

$$< p_{\text{loss}} > = \frac{1}{2} R_{\text{loss}} V^2 |Y(j\omega)|^2.$$

Consequently, because

$$|Y(j\omega)| = \frac{\sqrt{R^2 + \omega^2 L^2}}{R^2 + \omega^2 L^2} \geq \frac{R}{R^2 + \omega^2 L^2} = |Y_C(j\omega)|,$$

your power supplier will prefer to power a motor that behaves like $Y_C(j\omega)$ as compared to $Y(j\omega)$, since it can only charge for the power effectively delivered. The "power factor" $\cos(\angle(Y(j\omega))$ quantifies then a measure of efficiency for the power company, relating the power it delivers and charges, proportional to $|Y(j\omega)| \cos(\angle(Y(j\omega))$, with the power lost in the transmission, proportional to $|Y(j\omega)|^2$.