CHAPTER 8

Internal Flow
Having acquired the means to compute convection transfer rates for external flow, we now consider the convection transfer problem for internal flow. Recall that an external flow is one for which boundary layer development on a surface is allowed to continue without external constraints, as for the flat plate of Figure 6.6. In contrast, an internal flow, such as flow in a pipe, is one for which the fluid is confined by a surface. Hence the boundary layer is unable to develop without eventually being constrained. The internal flow configuration represents a convenient geometry for heating and cooling fluids used in chemical processing, environmental control, and energy conversion technologies.

Our objectives are to develop an appreciation for the physical phenomena associated with internal flow and to obtain convection coefficients for flow conditions of practical importance. We begin by considering velocity (hydrodynamic) effects pertinent to internal flows, focusing on certain unique features of boundary layer development. Thermal boundary layer effects are considered next, and an overall energy balance is applied to determine fluid temperature variations in the flow direction. Finally, correlations for estimating the convection heat transfer coefficient are presented for a variety of internal flow conditions.

### 3.1 Hydrodynamic Considerations

When considering external flow, it is necessary to ask only whether the flow is laminar or turbulent. However, for an internal flow we must also be concerned with the existence of entrance and fully developed regions.

#### 8.1.1 Flow Conditions

Consider laminar flow in a circular tube of radius \( r_c \) (Figure 8.1), where fluid enters the tube with a uniform velocity. We know that when the fluid makes contact with the surface, viscous effects become important, and a boundary layer develops with increasing \( x \). This development occurs at the expense of a shrinking inviscid flow.

![Figure 8.1 Laminar, hydrodynamic boundary layer development in a circular tube.](image-url)
region and concludes with boundary layer merger at the centerline. Following this merger, viscous effects extend over the entire cross section and the velocity profile no longer changes with increasing \( x \). The flow is then said to be fully developed, and the distance from the entrance at which this condition is achieved is termed the hydrodynamic entry length, \( x_{el} \). As shown in Figure 8.1, the fully developed velocity profile is parabolic for laminar flow in a circular tube. For turbulent flow, the profile is flatter due to turbulent mixing in the radial direction.

When dealing with internal flows, it is important to be cognizant of the extent of the entry region, which depends on whether the flow is laminar or turbulent. The Reynolds number for flow in a circular tube is defined as

\[
Re_D = \frac{\rho u_m D}{\mu} = \frac{u_m D}{v}
\]  

(8.1)

where \( u_m \) is the mean fluid velocity over the tube cross section and \( D \) is the tube diameter. In a fully developed flow, the critical Reynolds number corresponding to the onset of turbulence is

\[
Re_D = 2300
\]  

(8.2)

although much larger Reynolds numbers (\( Re_D \approx 10,000 \)) are needed to achieve fully turbulent conditions. The transition to turbulence is likely to begin in the developing boundary layer of the entrance region.

For laminar flow (\( Re_D \approx 2300 \)), the hydrodynamic entry length may be obtained from an expression of the form [1]

\[
\left( \frac{x_{el}}{D} \right)_{lam} = 0.05 Re_D
\]  

(8.3)

This expression is based on the presumption that fluid enters the tube from a rounded converging nozzle and is hence characterized by a nearly uniform velocity profile at the entrance (Figure 8.1). Although there is no satisfactory general expression for the entry length in turbulent flow, we know that it is approximately independent of Reynolds number and that, as a first approximation [2],

\[
10 \leq \left( \frac{x_{el}}{D} \right)_{turb} \leq 60
\]  

(8.4)

For the purposes of this text, we shall assume fully developed turbulent flow for \( (x/D) > 10 \).

### 8.1.2 The Mean Velocity

Because the velocity varies over the cross section and there is no well-defined free stream, it is necessary to work with a mean velocity \( u_m \) when dealing with internal flows. This velocity is defined such that, when multiplied by the fluid density \( \rho \) and the cross-sectional area of the tube \( A_c \), it provides the rate of mass flow through the tube. Hence

\[
\dot{m} = \rho u_m A_c
\]  

(8.5)
For steady, incompressible flow in a tube of uniform cross-sectional area, \( \dot{m} \) and \( u_m \) are constants independent of \( x \). From Equations 8.1 and 8.5 it is evident that, for flow in a circular tube \( (A_e = \pi D^2/4) \), the Reynolds number reduces to

\[
Re_D = \frac{4\dot{m}}{\pi D \mu}
\]  

(8.6)

Since the mass flow rate may also be expressed as the integral of the mass flux \( (\rho u) \) over the cross section

\[
\dot{m} = \int_{A_e} \rho u(r, x) \, dA_e
\]  

(8.7)

and it follows that, for incompressible flow in a circular tube,

\[
u_m = \frac{\int_{A_e} \rho u(r, x) \, dA_e}{\rho A_e} = \frac{2\pi \rho}{\rho \pi r^2} \int_0^{r_e} u(r, x) r \, dr = \frac{2}{r_e^2} \int_0^{r_e} u(r, x) r \, dr
\]  

(8.8)

The foregoing expression may be used to determine \( u_m \) at any axial location \( x \) from knowledge of the velocity profile \( u(r) \) at that location.

8.1.3 Velocity Profile in the Fully Developed Region

The form of the velocity profile may readily be determined for the laminar flow of an incompressible, constant property fluid in the fully developed region of a circular tube. An important feature of hydrodynamic conditions in the fully developed region is that both the radial velocity component \( v \) and the gradient of the axial velocity component \( \frac{\partial u}{\partial x} \) are everywhere zero.

\[
v = 0 \quad \text{and} \quad \left( \frac{\partial u}{\partial x} \right) = 0
\]  

(8.9)

Hence the axial velocity component depends only on \( r \), \( u(x, r) = u(r) \).

The radial dependence of the axial velocity may be obtained by solving the appropriate form of the \( x \)-momentum equation. This form is determined by first recognizing that, for the conditions of Equation 8.9, the net momentum flux is everywhere zero in the fully developed region. Hence the momentum conservation requirement reduces to a simple balance between shear and pressure forces in the flow. For the annular differential element of Figure 8.2, this force balance may be expressed as

\[
\tau_r(2\pi r \, dx) - \left[ \tau_r(2\pi r \, dx) + \frac{d}{dx}[\tau_r(2\pi r \, dx)] \right] dx
\]
\[
+ p(2\pi r \, dr) - \left[ p(2\pi r \, dr) + \frac{d}{dx}[p(2\pi r \, dr)] \right] dx = 0
\]

which reduces to

\[
- \frac{d}{dr} (r \tau_r) = r \frac{dp}{dx}
\]  

(8.10)
With \( y = r_o - r \), Newton's law of viscosity, Equation 6.8.10, assumes the form
\[
\tau_r = -\mu \frac{du}{dr}
\]
(8.11)
and Equation 8.10 becomes
\[
\frac{\mu}{r} \frac{d}{dr} \left( r \frac{du}{dr} \right) = \frac{dp}{dx}
\]
(8.12)
Since the axial pressure gradient is independent of \( r \), Equation 8.12 may be solved by integrating twice to obtain
\[
r \frac{du}{dr} = \frac{1}{\mu} \left( \frac{dp}{dx} \right) \frac{r^2}{2} + C_1
\]
and
\[
u(r) = \frac{1}{\mu} \left( \frac{dp}{dx} \right) \frac{r^2}{4} + C_1 \ln r + C_2
\]
The integration constants may be determined by invoking the boundary conditions
\[
u(r_o) = 0 \quad \text{and} \quad \left. \frac{\partial \nu}{\partial r} \right|_{r=r_o} = 0
\]
which, respectively, impose the requirements of zero slip at the tube surface and radial symmetry about the centerline. It is a simple matter to evaluate the constants, and it follows that
\[
u(r) = -\frac{1}{4\mu} \left( \frac{dp}{dx} \right) r^2 \left[ 1 - \left( \frac{r}{r_o} \right)^2 \right]
\]
(8.13)
Hence the fully developed velocity profile is parabolic. Note that the pressure gradient must always be negative.
The foregoing result may be used to determine the mean velocity of the flow. Substituting Equation 8.13 into Equation 8.8 and integrating, we obtain
\[
u_m = -\frac{r_o^2}{8\mu} \frac{dp}{dx}
\]
(8.14)
Substituting this result into Equation 8.13, the velocity profile is then

\[ \frac{u(r)}{u_m} = 2 \left[ 1 - \left( \frac{r}{r_o} \right)^2 \right] \]  

(Equation 8.15)

Since \( u_m \) can be computed from knowledge of the mass flow rate, Equation 8.14 can be used to determine the pressure gradient.

### 8.1.4 Pressure Gradient and Friction Factor in Fully Developed Flow

The engineer is frequently interested in the pressure drop needed to sustain an internal flow because this parameter determines pump or fan power requirements. To determine the pressure drop, it is convenient to work with the Moody (or Darcy) friction factor, which is a dimensionless parameter defined as

\[ f = \frac{-(dp/dx)D}{\rho u_m^2/2} \]  

(Equation 8.16)

This quantity is not to be confused with the friction coefficient, sometimes called the Fanning friction factor, which is defined as

\[ C_f = \frac{\tau_w}{\rho u_m^2/2} \]  

(Equation 8.17)

Since \( \tau_w = -\mu (du/dr)_{r=r_m} \), it follows from Equation 8.13 that

\[ C_f = \frac{f}{4} \]  

(Equation 8.18)

Substituting Equations 8.1 and 8.14 into 8.16, it follows that, for fully developed laminar flow,

\[ f = \frac{64}{Re_D} \]  

(Equation 8.19)

For fully developed turbulent flow, the analysis is much more complicated, and we must ultimately rely on experimental results. Friction factors for a wide Reynolds number range are presented in the Moody diagram of Figure 8.3. In addition to depending on the Reynolds number, the friction factor is a function of the tube surface condition. It is a minimum for smooth surfaces and increases with increasing surface roughness, \( e \). Correlations that reasonably approximate the smooth surface condition are of the form

\[ f = 0.316 Re_D^{1/4} \quad Re_D \leq 2 \times 10^4 \]  

(Equation 8.20a)

\[ f = 0.184 Re_D^{1/5} \quad Re_D \approx 2 \times 10^4 \]  

(Equation 8.20b)

Alternatively, a single correlation that encompasses a large Reynolds number range has been developed by Petukhov [4] and is of the form

\[ f = (0.790 \ln Re_D - 1.64)^{-2} \quad 3000 \leq Re_D \leq 5 \times 10^6 \]  

(Equation 8.21)
8.2 Thermal Considerations

Note that $f$, hence $dp/dx$, is a constant in the fully developed region. From Equation 8.16 the pressure drop $\Delta p = p_1 - p_2$ associated with fully developed flow from the axial position $x_1$ to $x_2$ may then be expressed as

$$\Delta p = -\int_{p_1}^{p_2} dp = f\frac{\rho u_1^2}{2D} \int_{x_1}^{x_2} dx = f\frac{\rho u_1^2}{2D} (x_2 - x_1)$$

(8.22a)

where $f$ is obtained from Figure 8.3 or from Equation 8.19 for laminar flow and from Equation 8.20 or 8.21 for turbulent flow in smooth tubes. The pump or fan power required to overcome the resistance to flow associated with this pressure drop may be expressed as

$$P = (\Delta p)\dot{V}$$

(8.22b)

where the volumetric flow rate $\dot{V}$ may, in turn, be expressed as $\dot{V} = \dot{m}/\rho$ for an incompressible fluid.

8.2 Thermal Considerations

Having reviewed the fluid mechanics of internal flow, we now consider thermal effects. If fluid enters the tube of Figure 8.4 at a uniform temperature $T(r, 0)$ that is
less than the surface temperature, convection heat transfer occurs and a thermal boundary layer begins to develop. Moreover, if the tube surface condition is fixed by imposing either a uniform temperature (\(T_s\) is constant) or a uniform heat flux (\(q'_f\) is constant), a thermally fully developed condition is eventually reached. The shape of the fully developed temperature profile \(T(r,x)\) differs according to whether a uniform surface temperature or heat flux is maintained. For both surface conditions, however, the amount by which fluid temperatures exceed the entrance temperature increases with increasing \(x\).

For laminar flow the thermal entry length may be expressed as [2]

\[
\frac{x_{el,t}}{D} = 0.05 Re_d Pr
\]  

(8.23)

Comparing Equations 8.3 and 8.23, it is evident that, if \(Pr > 1\), the hydrodynamic boundary layer develops more rapidly than the thermal boundary layer (\(x_{el,t} < x_{el,h}\)), while the inverse is true for \(Pr < 1\). For extremely large Prandtl number fluids, such as oils (\(Pr \approx 100\)), \(x_{el,t}\) is very much smaller than \(x_{el,h}\) and it is reasonable to assume a fully developed velocity profile throughout the thermal entry region. In contrast, for turbulent flow, conditions are nearly independent of Prandtl number, and to a first approximation, we shall assume \((x_{el,t}/D) = 10\).

Thermal conditions in the fully developed region are characterized by several interesting and useful features. Before we can consider these features (Section 8.2.3), however, it is necessary to introduce the concept of a mean temperature and the appropriate form of Newton’s law of cooling.

### 8.2.1 The Mean Temperature

Just as the absence of a free stream velocity requires use of a mean velocity to describe an internal flow, the absence of a fixed free stream temperature necessitates using a mean (or bulk) temperature. To provide a definition of the mean temperature, we begin by returning to Equation 1.11e:

\[
q = \dot{m} c_p (T_{out} - T_{in})
\]  

(1.11e)
Recall that the terms on the right-hand side represent the thermal energy for an incompressible liquid or the enthalpy (thermal energy plus flow work) for an ideal gas, which is carried by the fluid. In developing this equation, it was implicitly assumed that the temperature was uniform across the inlet and outlet cross-sectional areas. In reality, this is not true if convection heat transfer occurs, and we define the mean temperature so that the term \( \dot{m}c_p T_m \) is equal to the true rate of thermal energy (or enthalpy) advection integrated over the cross section. This true advection rate may be obtained by integrating the product of mass flux (\( \dot{m} \)) and the thermal energy (or enthalpy) per unit mass, \( c_p T \), over the cross section. Therefore, we define \( T_m \) from

\[
\dot{m}c_p T_m = \int_A \rho u c_p T \, dA_c \tag{8.24}
\]

or

\[
T_m = \frac{\int_A \rho u c_p T \, dA_c}{\dot{m}c_p} \tag{8.25}
\]

For flow in a circular tube with constant \( \rho \) and \( c_p \), it follows from Equations 8.5 and 8.25 that

\[
T_m = \frac{2}{u_0^2 \rho c_p} \int_0^r u T \, dr \tag{8.26}
\]

It is important to note that, when multiplied by the mass flow rate and the specific heat, \( T_m \) provides the rate at which thermal energy (or enthalpy) is advected with the fluid as it moves along the tube.

### 8.2.2 Newton's Law of Cooling

The mean temperature \( T_m \) is a convenient reference temperature for internal flows, playing much the same role as the free stream temperature \( T_w \) for external flows. Accordingly, Newton's law of cooling may be expressed as

\[
q''_s = h(T_s - T_m) \tag{8.27}
\]

where \( h \) is the local convection heat transfer coefficient. However, there is an essential difference between \( T_m \) and \( T_w \). Whereas \( T_w \) is constant in the flow direction, \( T_m \) must vary in this direction. That is, \( dT_m/dx \) is never zero if heat transfer is occurring. The value of \( T_m \) increases with \( x \) if heat transfer is from the surface to the fluid (\( T_s > T_m \)); it decreases with \( x \) if the opposite is true (\( T_s < T_m \)).

### 8.2.3 Fully Developed Conditions

Since the existence of convection heat transfer between the surface and the fluid dictates that the fluid temperature must continue to change with \( x \), one might legitimately question whether fully developed thermal conditions can ever be reached. The situation is certainly different from the hydrodynamic case, for which \( (\partial u/\partial x) = 0 \) in the fully developed region. In contrast, if there is heat transfer, \( (dT_s/\partial x) \), as well as \( (\partial T/\partial x) \) at any radius \( r \), is not zero. Accordingly, the temperature profile \( T(r) \) is
continuously changing with \( x \), and it would seem that a fully developed condition could never be reached. This apparent contradiction may be reconciled by working with a dimensionless form of the temperature.

Analyses may be simplified by working with dimensionless temperature differences, as for transient conduction (Chapter 5) and the energy conservation equation (Chapter 6). Introducing a dimensionless temperature difference of the form \( (T_s - T) / (T_s - T_m) \), conditions for which this ratio becomes independent of \( x \) are known to exist [2]. That is, although the temperature profile \( T(r) \) continues to change with \( x \), the relative shape of the profile no longer changes and the flow is said to be thermally fully developed. The requirement for such a condition is formally stated as

\[
\frac{\partial}{\partial x} \left[ \frac{T_s(x) - T(r, x)}{T_s(x) - T_m(x)} \right] = 0
\]

where \( T_s \) is the tube surface temperature, \( T \) is the local fluid temperature, and \( T_m \) is the mean temperature of the fluid over the cross section of the tube.

The condition given by Equation 8.28 is eventually reached in a tube for which there is either a uniform surface heat flux \( (q_s^u \) is constant) or a uniform surface temperature \( (T_s \) is constant). These surface conditions arise in many engineering applications. For example, a constant surface heat flux would exist if the tube wall were heated electrically or if the outer surface were uniformly irradiated. In contrast, a constant surface temperature would exist if a phase change (due to boiling or condensation) were occurring at the outer surface. Note that it is impossible to simultaneously impose the conditions of constant surface heat flux and constant surface temperature. If \( q_s^u \) is constant, \( T_s \) must vary with \( x \); conversely, if \( T_s \) is constant, \( q_s^u \) must vary with \( x \).

Several important features of thermally developed flow may be inferred from Equation 8.28. Since the temperature ratio is independent of \( x \), the derivative of this ratio with respect to \( r \) must also be independent of \( x \). Evaluating this derivative at the tube surface (note that \( T_s \) and \( T_m \) are constants insofar as differentiation with respect to \( r \) is concerned), we then obtain

\[
\left. \frac{\partial}{\partial r} \left( \frac{T_s - T}{T_s - T_m} \right) \right|_{r=r_s} = \frac{-\partial T_s/\partial r}{T_s - T_m} \neq f(x)
\]

Substituting for \( \partial T_s/\partial r \) from Fourier's law, which, from Figure 8.4, is of the form

\[
q_s^u = -k \frac{\partial T}{\partial y} \bigg|_{r=r_s} = k \frac{\partial T}{\partial r} \bigg|_{r=r_s}
\]

and for \( q_s^u \) from Newton's law of cooling, Equation 8.27, we obtain

\[
\frac{h}{k} \neq f(x)
\]

Hence in the thermally fully developed flow of a fluid with constant properties, the local convection coefficient is a constant, independent of \( x \).

Equation 8.28 is not satisfied in the entrance region, where \( h \) varies with \( x \), as shown in Figure 8.5. Because the thermal boundary layer thickness is zero at the tube entrance, the convection coefficient is extremely large at \( x = 0 \). However, \( h \) decays rapidly as the thermal boundary layer develops, until the constant value associated with fully developed conditions is reached.
Additional simplifications are associated with the special case of uniform surface heat flux. Since both \( h \) and \( q'' \) are constant in the fully developed region, it follows from Equation 8.27 that

\[
\frac{dT_s}{dx} \bigg|_{x_{1a}} = \frac{dT_m}{dx} \bigg|_{x_{1a}} \quad q'' = \text{constant} \quad (8.30)
\]

If we expand Equation 8.28 and solve for \( \frac{dT}{dx} \), it also follows that

\[
\frac{dT}{dx} \bigg|_{x_{1a}} = \frac{dT_s}{dx} \bigg|_{x_{1a}} - \frac{(T_s - T)}{(T_s - T_m)} \frac{dT_s}{dx} \bigg|_{x_{1a}} + \frac{(T_s - T)}{(T_s - T_m)} \frac{dT_m}{dx} \bigg|_{x_{1a}} \quad (8.31)
\]

Substituting from Equation 8.30, we then obtain

\[
\frac{dT}{dx} \bigg|_{x_{1a}} = \frac{dT_m}{dx} \bigg|_{x_{1a}} \quad q'' = \text{constant} \quad (8.32)
\]

Hence the axial temperature gradient is independent of the radial location. For the case of constant surface temperature \( (dT_s/dx = 0) \), it also follows from Equation 8.31 that

\[
\frac{dT}{dx} \bigg|_{x_{1a}} = \frac{(T_s - T)}{(T_s - T_m)} \frac{dT_m}{dx} \bigg|_{x_{1a}} \quad T_s = \text{constant} \quad (8.33)
\]

in which case the value of \( \partial T/\partial x \) depends on the radial coordinate.

From the foregoing results, it is evident that the mean temperature is a very important variable for internal flows. To describe such flows, its variation with \( x \) must be known. This variation may be obtained by applying an overall energy balance to the flow, as will be shown in the next section.

**Example 8.1**

For flow of a liquid metal through a circular tube, the velocity and temperature profiles at a particular axial location may be approximated as being uniform and parabolic, respectively. That is, \( u(r) = \frac{C_1}{r} \) and \( T(r) = T_s - C_2(1 - (r/r_c)^2) \), where \( C_1 \) and \( C_2 \) are constants. What is the value of the Nusselt number \( Nu_D \) at this location?
SOLUTION

**Known:** Form of the velocity and temperature profiles at a particular axial location for flow in a circular tube.

**Find:** Nusselt number at the prescribed location.

**Schematic:**

![Schematic diagram](image)

**Assumptions:** Incompressible, constant property flow.

**Analysis:** The Nusselt number may be obtained by first determining the convection coefficient, which, from Equation 8.27, is given as

$$h = \frac{q''_r}{T_s - T_m}$$

From Equation 8.26, the mean temperature is

$$T_m = \frac{2}{u_m r_o^2} \int_0^r u T r \, dr = \frac{2C_1}{u_m r_o^2} \int_0^r \left( T_s + C_2 \left[ 1 - \left( \frac{r}{r_o} \right)^2 \right] \right) r \, dr$$

or, since $u_m = C_1$ from Equation 8.8,

$$T_m = \frac{2}{r_o^2} \int_0^r \left( T_s + C_2 \left[ 1 - \left( \frac{r}{r_o} \right)^2 \right] \right) r \, dr$$

$$T_m = \frac{2}{r_o^2} \left[ T_s r_o^2 + C_2 \left( \frac{r_o^2}{2} - \frac{C_2 r_o^2}{4} \right) \right]_0$$

$$T_m = \frac{2}{r_o^2} \left( T_s r_o^2 + C_2 r_o^2 - \frac{C_2 r_o^2}{4} \right) = T_s + \frac{C_2}{2}$$

The heat flux may be obtained from Fourier's law, in which case

$$q''_r = k \frac{\partial T}{\partial r} \bigg|_{r=r_o} = -k C_2 \left( \frac{r}{r_o} \right) \bigg|_{r=r_o} = -2C_2 \frac{k}{r_o}$$

Hence

$$h = \frac{q''_r}{T_s - T_m} = \frac{-2C_2(k/r_o)}{-C_2} = \frac{4k}{r_o}$$
8.3 The Energy Balance

8.3.1 General Considerations

Because the flow in a tube is completely enclosed, an energy balance may be applied to determine how the mean temperature \( T_m(x) \) varies with position along the tube and how the total convection heat transfer \( q_{\text{conv}} \) is related to the difference in temperatures at the tube inlet and outlet. Consider the tube flow of Figure 8.6. Fluid moves at a constant flow rate \( \dot{m} \), and convection heat transfer occurs at the inner surface. Typically, it will be reasonable to make one of the four assumptions in Section 1.3 that leads to the simplified steady-flow thermal energy equation, Equation 1.11e. For example, it is often the case that viscous dissipation is negligible (see Problem 8.10) and that the fluid can be modeled as either an incompressible liquid or an ideal gas with negligible pressure variation. In addition, it is usually reasonable to neglect heat transfer by conduction in the axial direction, so the heat transfer term in Equation 1.11e includes only \( q_{\text{conv}} \). Therefore, Equation 1.11e may be written in the form

\[
q_{\text{conv}} = \dot{m}c_p(T_{m,o} - T_{m,i})
\]  

(8.34)

for a tube of finite length. This simple overall energy balance relates three important thermal variables \( q_{\text{conv}}, T_{m,o}, T_{m,i} \). It is a general expression that applies irrespective of the nature of the surface thermal or tube flow conditions.

Applying Equation 1.11e to the differential control volume of Figure 8.6 and recalling that the mean temperature is defined such that \( \dot{m}c_p T_m \) represents the true rate of thermal energy (or enthalpy) advection integrated over the cross-section, we obtain

\[
dq_{\text{conv}} = \dot{m}c_p(T_m + dT_m) - T_m
\]  

(8.35)

\[
dq_{\text{conv}} = q_{\text{conv}}^p dx
\]

**Figure 8.6** Control volume for internal flow in a tube.
or

\[ dq_{\text{conv}} = \dot{m}c_p dT_m \]  

(8.36)

Equation 8.36 may be cast in a convenient form by expressing the rate of convection heat transfer to the differential element as \( dq_{\text{conv}} = \frac{q''_m P}{dx} \), where \( P \) is the surface perimeter (\( P = \pi D \) for a circular tube). Substituting from Equation 8.27, it follows that

\[ \frac{dT_m}{dx} = \frac{q''_m P}{\dot{m}c_p} = \frac{P}{\dot{m}c_p} h(T_s - T_m) \]  

(8.37)

This expression is an extremely useful result, from which the axial variation of \( T_m \) may be determined. If \( T_s > T_m \), heat is transferred to the fluid and \( T_m \) increases with \( x \); if \( T_s < T_m \), the opposite is true.

The manner in which quantities on the right-hand side of Equation 8.37 vary with \( x \) should be noted. Although \( P \) may vary with \( x \), most commonly it is a constant (a tube of constant cross-sectional area). Hence the quantity \( (P/\dot{m}c_p) \) is a constant. In the fully developed region, the convection coefficient \( h \) is also constant, although it varies with \( x \) in the entrance region (Figure 8.5). Finally, although \( T_s \) may be constant, \( T_m \) must always vary with \( x \) (except for the trivial case of no heat transfer, \( T_s = T_m \)).

The solution to Equation 8.37 for \( T_m(x) \) depends on the surface thermal condition. Recall that the two special cases of interest are constant surface heat flux and constant surface temperature. It is common to find one of these conditions existing to a reasonable approximation.

### 8.3.2 Constant Surface Heat Flux

For constant surface heat flux we first note that it is a simple matter to determine the total heat transfer rate \( q_{\text{conv}} \). Since \( q''_m \) is independent of \( x \), it follows that

\[ q_{\text{conv}} = q''_m (P \cdot L) \]  

(8.38)

This expression could be used with Equation 8.34 to determine the fluid temperature change, \( T_{m,i} - T_{m,f} \).

For constant \( q''_m \) it also follows that the middle expression in Equation 8.37 is a constant independent of \( x \). Hence

\[ \frac{dT_m}{dx} = \frac{q''_m P}{\dot{m}c_p} \neq f(x) \]  

(8.39)

Integrating from \( x = 0 \), it follows that

\[ T_m(x) = T_{m,i} + \frac{q''_m P}{\dot{m}c_p} x \]  

(8.40)

Accordingly, the mean temperature varies linearly with \( x \) along the tube (Figure 8.7a). Moreover, from Equation 8.27 and Figure 8.5 we also expect the temperature difference \( (T_s - T_m) \) to vary with \( x \), as shown in Figure 8.7a. This difference is initially...
Figure 8.7 Axial temperature variations for heat transfer in a tube. (a) Constant surface heat flux. (b) Constant surface temperature.

small (due to the large value of $h$ near the entrance) but increases with increasing $x$ due to the decrease in $h$ that occurs as the boundary layer develops. However, in the fully developed region we know that $h$ is independent of $x$. Hence from Equation 8.27 it follows that $(T_i - T_m)$ must also be independent of $x$ in this region.

It should be noted that, if the heat flux is not constant but is, instead, a known function of $x$, Equation 8.37 may still be integrated to obtain the variation of the mean temperature with $x$. Similarly, the total heat rate may be obtained from the requirement that $q_{conv} = \int_0^L q'(x) dx$.

Example 8.2

A system for heating water from an inlet temperature of $T_{i,n} = 20^\circ C$ to an outlet temperature of $T_{o,n} = 60^\circ C$ involves passing the water through a thick-walled tube having inner and outer diameters of 20 and 40 mm. The outer surface of the tube is well insulated, and electrical heating within the wall provides for a uniform generation rate of $\dot{q} = 10^6$ W/m$^3$.

1. For a water mass flow rate of $\dot{m} = 0.1$ kg/s, how long must the tube be to achieve the desired outlet temperature?

2. If the inner surface temperature of the tube is $T_i = 70^\circ C$ at the outlet, what is the local convection heat transfer coefficient at the outlet?

Solution

Known: Internal flow through thick-walled tube having uniform heat generation.

Find:

1. Length of tube needed to achieve the desired outlet temperature.

2. Local convection coefficient at the outlet.
Schematic:

Assumptions:
1. Steady-state conditions.
2. Uniform heat flux.
3. Incompressible liquid and negligible viscous dissipation.
4. Constant properties.
5. Adiabatic outer tube surface.

Properties: Table A.6, water \((\bar{T}_w = 313 \text{ K})\): \(c_p = 4179 \text{ J/kg \cdot K}\).

Analysis:
1. Since the outer surface of the tube is adiabatic, the rate at which energy is generated within the tube wall must equal the rate at which it is convected to the water.

\[
\dot{E}_g = \dot{q}_{conv}
\]

With

\[
\dot{E}_g = \dot{q} \frac{\pi}{4} (D_o^2 - D_i^2)L
\]

it follows from Equation 8.34 that

\[
\dot{q} \frac{\pi}{4} (D_o^2 - D_i^2)L = \dot{m}c_p(T_{m,o} - T_{m,i})
\]

or

\[
L = \frac{4\dot{m}c_p}{\pi(D_o^2 - D_i^2)q} (T_{m,o} - T_{m,i})
\]

\[
L = \frac{4 \times 0.1 \text{ kg/s} \times 4179 \text{ J/kg \cdot K}}{\pi(0.04^2 - 0.02^2)} \text{ m}^2 \times \frac{10^6 \text{ W/m}^2}{(60 - 20) \text{ K}} = 17.7 \text{ m}
\]

2. From Newton’s law of cooling, Equation 8.27, the local convection coefficient at the tube exit is

\[
h_o = \frac{\dot{q}}{T_{s,o} - T_{m,o}}
\]
Assuming that uniform heat generation in the wall provides a constant surface heat flux, with

\[
q''_w = \frac{E_s}{\pi D_i L} = \frac{\dot{q}}{\pi D_i L} = \frac{D_o^2 - D_i^2}{4 D_i}
\]

\[
q''_w = \frac{10^6 \text{W/m}^3}{4} \left( \frac{(0.04^2 - 0.02^2)}{0.02 \text{m}} \right) = 1.5 \times 10^4 \text{W/m}^3
\]

it follows that

\[
h_w = \frac{1.5 \times 10^4 \text{W/m}^3}{(70 - 60) \text{^\circ C}} = 1500 \text{ W/m}^2 \cdot \text{K}
\]

Comments:
1. If conditions are fully developed over the entire tube, the local convection coefficient and the temperature difference \((T_s - T_w)\) are independent of \(x\). Hence \(h = 1500 \text{ W/m}^2 \cdot \text{K}\) and \((T_s - T_w) = 10 \text{^\circ C}\) over the entire tube. The inner surface temperature at the tube inlet is then \(T_{si} = 30 \text{^\circ C}\).

2. The required tube length \(L\) could have been computed by applying the expression for \(T_w(x)\), Equation 8.40, at \(x = L\).

8.3.3 Constant Surface Temperature

Results for the total heat transfer rate and the axial distribution of the mean temperature are entirely different for the constant surface temperature condition. Defining \(\Delta T\) as \(T_s - T_w\), Equation 8.37 may be expressed as

\[
\frac{dT_w}{dx} = -\frac{d(\Delta T)}{dx} = \frac{P}{mc_p} \Delta T
\]

Separating variables and integrating from the tube inlet to the outlet,

\[
\int_{\Delta T_s}^{\Delta T_w} \frac{d(\Delta T)}{\Delta T} = -\frac{P}{mc_p} \int_0^L h \, dx
\]

or

\[
\ln \frac{\Delta T_w}{\Delta T_s} = -\frac{PL}{mc_p} \left( \frac{1}{L} \int_0^L h \, dx \right)
\]

From the definition of the average convection heat transfer coefficient, Equation 6.9, it follows that

\[
\ln \frac{\Delta T_w}{\Delta T_s} = -\frac{PL}{mc_p} \bar{h}_L, \quad T_s = \text{constant}
\]

\[\text{(8.41a)}\]
where $\bar{h}_L$, or simply $\bar{h}$, is the average value of $h$ for the entire tube. Rearranging,

$$\frac{\Delta T_o}{\Delta T_i} = \frac{T_0 - T_{m,i}}{T_i - T_{m,i}} = \exp \left( -\frac{PL}{mc_p \bar{h}} \right) \quad T_i = \text{constant} \tag{8.41b}$$

Had we integrated from the tube inlet to some axial position $x$ within the tube, we would have obtained the similar, but more general, result that

$$\frac{T_i - T_{m_i}(x)}{T_i - T_{m,i}} = \exp \left( -\frac{P_x}{mc_p \bar{h}} \right) \quad T_i = \text{constant} \tag{8.42}$$

where $\bar{h}$ is now the average value of $h$ from the tube inlet to $x$. This result tells us that the temperature difference $(T_i - T_{m,i})$ decays exponentially with distance along the tube axis. The axial surface and mean temperature distributions are therefore as shown in Figure 8.7b.

Determination of an expression for the total heat transfer rate $q_{\text{conv}}$ is complicated by the exponential nature of the temperature decay. Expressing Equation 8.34 in the form

$$q_{\text{conv}} = \dot{m}c_p [(T_s - T_{m,s}) - (T_i - T_{m,i})] = \dot{m}c_p (\Delta T_i - \Delta T_o)$$

and substituting for $\dot{m}c_p$ from Equation 8.41a, we obtain

$$q_{\text{conv}} = \bar{h}A_i \Delta T_{\text{lm}} \quad T_i = \text{constant} \tag{8.43}$$

where $A_i$ is the tube surface area ($A_i = P \cdot L$) and $\Delta T_{\text{lm}}$ is the log mean temperature difference,

$$\Delta T_{\text{lm}} = \frac{\Delta T_o - \Delta T_i}{\ln (\Delta T_o/\Delta T_i)} \tag{8.44}$$

Equation 8.43 is a form of Newton's law of cooling for the entire tube, and $\Delta T_{\text{lm}}$ is the appropriate average of the temperature difference over the tube length. The logarithmic nature of this average temperature difference [in contrast, e.g., to an arithmetic mean temperature difference of the form $\Delta T_{\text{lm}} = (\Delta T_i + \Delta T_o)/2$] is due to the exponential nature of the temperature decay.

Before concluding this section, it is important to note that, in many applications, it is the temperature of an external fluid, rather than the tube surface temperature, that is fixed (Figure 8.8). In such cases, it is readily shown that the results of this section may still be used if $T_i$ is replaced by $T_e$ (the free stream temperature of the external fluid) and $h$ is replaced by $\bar{U}$ (the average overall heat transfer coefficient). For such cases, it follows that

$$\frac{\Delta T_o}{\Delta T_i} = \frac{T_0 - T_{m,i}}{T_e - T_{m,i}} = \exp \left( \frac{UL}{mc_p \bar{U}} \right) \tag{8.45a}$$

and

$$q = \bar{U}A_i \Delta T_{\text{lm}} \tag{8.46a}$$
The overall heat transfer coefficient is defined in Section 3.3.1, and for this application it would include contributions due to convection at the tube inner and outer surfaces. For a thick-walled tube of small thermal conductivity, it would also include the effect of conduction across the tube wall. Note that the product $UA$ yields the same result, irrespective of whether it is defined in terms of the inner ($UA_i$) or outer ($UA_o$) surface areas of the tube (see Equation 3.32). Also note that $(UA)^{-1}$ is equivalent to the total thermal resistance between the two fluids, in which case Equations 8.45a and 8.46a may be expressed as

$$\frac{\Delta T_o}{\Delta T_i} = \frac{T_m - T_{m,o}}{T_o - T_{m,i}} = \exp \left( -\frac{1}{mc_p R_{int}} \right)$$  \hspace{1cm} (8.45b)

and

$$q = \frac{\Delta T_{hi}}{R_{tot}}$$ \hspace{1cm} (8.46b)

A common variation of the foregoing conditions is one for which the uniform temperature of an outer surface, $T_{s,o}$, rather than the free stream temperature of an external fluid, $T_o$, is known. In the foregoing equations, $T_m$ is then replaced by $T_{s,o}$, and the total resistance embodies the convection resistance associated with the internal flow, as well as the total resistance due to conduction between the inner surface of the tube and the surface corresponding to $T_{s,o}$.

**Example 8.3**

Steam condensing on the outer surface of a thin-walled circular tube of diameter $D = 50$ mm and length $L = 6$ m maintains a uniform outer surface temperature of 100°C. Water flows through the tube at a rate of $m = 0.25$ kg/s, and its inlet and outlet temperatures are $T_{m,i} = 15$°C and $T_{m,o} = 57$°C. What is the average convection coefficient associated with the water flow?
**SOLUTION**

**Known:** Flow rate and inlet and outlet temperatures of water flowing through a tube of prescribed dimensions and surface temperature.

**Find:** Average convection heat transfer coefficient.

**Schematic:**

![Diagram of a tube with temperatures and dimensions](image)

**Assumptions:**
1. Negligible tube wall conduction resistance.
2. Incompressible liquid and negligible viscous dissipation.
3. Constant properties.

**Properties:** Table A.6, water (36°C): $c_p = 4178$ J/kg · K.

**Analysis:** Combining the energy balance, Equation 8.34, with the rate equation, Equation 8.43, the average convection coefficient is given by

$$\bar{h} = \frac{\dot{m} c_p (T_{in,o} - T_{m,i})}{\pi DL \Delta T_{lm}}$$

From Equation 8.44

$$\Delta T_{lm} = \frac{(T_i - T_{m,o}) - (T_i - T_{m,o})}{\ln [(T_i - T_{m,o})/(T_i - T_{m,o})]}$$

$$\Delta T_{lm} = \frac{(100 - 57) - (100 - 15)}{\ln[(100 - 57)/(100 - 15)]} = 61.6°C$$

Hence

$$\bar{h} = \frac{0.25 \text{ kg/s} \times 4178 \text{ J/kg} \cdot \text{K} (57 - 15)°\text{C}}{\pi \times 0.05 \text{ m} \times 6 \text{ m} \times 61.6°C}$$

or

$$\bar{h} = 755 \text{ W/m}^2 \cdot \text{K}$$

**Comments:** If conditions were fully developed over the entire tube, the local convection coefficient would be everywhere equal to 755 W/m² · K.
8.4 Laminar Flow in Circular Tubes

8.4.1 Laminar Flow in Circular Tubes: Thermal Analysis and Convection Correlations

To use many of the foregoing results, the convection coefficients must be known. In this section we outline the manner in which such coefficients may be obtained theoretically for laminar flow in a circular tube. In subsequent sections we consider empirical correlations pertinent to turbulent flow in a circular tube, as well as to flows in tubes of noncircular cross section.

8.4.1 The Fully Developed Region

Here, the problem of heat transfer in laminar flow of an incompressible, constant property fluid in the fully developed region of a circular tube is treated theoretically. The resulting temperature distribution is used to determine the convection coefficient.

A differential equation governing the temperature distribution is determined by applying the simplified, steady-flow, thermal energy equation, Equation 1.11e \([q = mc_p (T_{ox} - T_{in})]\), to the annular differential element of Figure 8.9. If we neglect the effects of net axial conduction, the heat input, \(q_r\), is due only to conduction through the radial surfaces. Since the radial velocity is zero in the fully developed region, there is no advection of thermal energy through the radial control surfaces, and the only advection is in the axial direction. Thus, Equation 1.11e leads to Equation 8.47, which expresses a balance between radial conduction and axial advection:

\[
q_r - q_{r+dr} = (\dot{m}c_p) \left[ \left( T + \frac{\partial T}{\partial x} dx \right) - T \right]
\]  
\[
(\dot{m}c_p) \frac{\partial T}{\partial x} dx = q_r - \left( q_r + \frac{\partial q_r}{\partial r} dr \right) = -\frac{\partial q_r}{\partial r} dr
\]

\[
(\dot{m}c_p) \frac{\partial T}{\partial x} dx = q_r - \left( \frac{\partial q_r}{\partial r} dr \right) = -\frac{\partial q_r}{\partial r} dr
\]

\[
q_r = \left( \frac{\partial q_r}{\partial r} dr \right)
\]

\[
q_r = -\frac{\partial q_r}{\partial r} dr
\]

Figure 8.9 Thermal energy balance on a differential element for laminar, fully developed flow in a circular tube.
Chapter 8 - Internal Flow

The differential mass flow rate in the axial direction is \( dm = \rho u 2\pi r dr \), and the radial heat transfer rate is \( q_r = -k (\partial T/\partial r) 2\pi r dr \). If we assume constant properties, Equation 8.47b becomes

\[
\frac{\partial T}{\partial x} = \frac{\alpha}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right)
\]  

(8.48)

We will now proceed to solve for the temperature distribution for the case of constant surface heat flux. In this case, the assumption of negligible net axial conduction is exactly satisfied, that is, \( (\partial^2 T/\partial x^2) = 0 \). Substituting for the axial temperature gradient from Equation 8.32 and for the axial velocity component, \( u \), from Equation 8.15, the energy equation, Equation 8.48, reduces to

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) = \frac{2\mu_m}{\alpha} \left( \frac{dT_m}{dx} \right) \left[ 1 - \left( \frac{r}{r_o} \right)^2 \right] q''_r = \text{constant}
\]  

(8.49)

where \( T_m(x) \) varies linearly with \( x \) and \( (2\mu_m/\alpha)(dT_m/dx) \) is a constant. Separating variables and integrating twice, we obtain an expression for the radial temperature distribution:

\[
T(r,x) = \frac{2\mu_m}{\alpha} \left( \frac{dT_m}{dx} \right) \left[ \frac{r^2}{4} - \frac{r^4}{16r_o^2} \right] + C_1 \ln r + C_2
\]

The constants of integration may be evaluated by applying appropriate boundary conditions. From the requirement that the temperature remain finite at \( r = 0 \), it follows that \( C_1 = 0 \). From the requirement that \( T(r_o) = T_s \), where \( T_s \) varies with \( x \), it also follows that

\[
C_2 = T_s(x) - \frac{2\mu_m}{\alpha} \left( \frac{dT_m}{dx} \right) \left( \frac{3r_o^2}{16} \right)
\]

Accordingly, for the fully developed region with constant surface heat flux, the temperature profile is of the form

\[
T(r,x) = T_s(x) - \frac{2\mu_m r_o^2}{\alpha} \left( \frac{dT_m}{dx} \right) \left[ \frac{3}{16} + \frac{1}{16} \left( \frac{r}{r_o} \right)^4 - \frac{1}{4} \left( \frac{r}{r_o} \right)^2 \right]
\]  

(8.50)

From knowledge of the temperature profile, all other thermal parameters may be determined. For example, if the velocity and temperature profiles, Equations 8.15 and 8.50, respectively, are substituted into Equation 8.26 and the integration over \( r \) is performed, the mean temperature is found to be

\[
T_m(x) = T_s(x) - \frac{11}{48} \left( \frac{\alpha u''_m}{2} \right) \left( \frac{dT_m}{dx} \right)
\]  

(8.51)

From Equation 8.39, where \( P = \pi D \) and \( \dot{m} = \rho u_m (\pi D^2)/4 \), we then obtain

\[
T_m(x) - T_s(x) = -\frac{11}{48} \frac{\dot{m} D}{k} \frac{q''_s}{k}
\]  

(8.52)

Combining Newton's law of cooling, Equation 8.27, and Equation 8.52, it follows that

\[
h = \frac{48}{11} \frac{k}{D}
\]
Hence in a circular tube characterized by uniform surface heat flux and laminar, fully developed conditions, the Nusselt number is a constant, independent of $Re_D$, $Pr$, and axial location.

For laminar, fully developed conditions with a constant surface temperature, the assumption of negligible axial conduction is often reasonable. Substituting for the velocity profile from Equation 8.15 and for the axial temperature gradient from Equation 8.33, the energy equation becomes

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) = \frac{2u_m}{\alpha} \left( \frac{dT_m}{dx} \right) \left[ 1 - \left( \frac{r}{r_m} \right)^2 \right] \frac{T_s - T}{T_s - T_m} \quad T_s = \text{constant} \quad (8.54)
\]

A solution to this equation may be obtained by an iterative procedure, which involves making successive approximations to the temperature profile. The resulting profile is not described by a simple algebraic expression, but the resulting Nusselt number may be shown to be \[2]\]

\[
Nu_D = 3.66 \quad T_s = \text{constant} \quad (8.55)
\]

Note that in using Equation 8.53 or 8.55 to determine $h$, the thermal conductivity should be evaluated at $T_m$.

Example 8.4

One concept used for solar energy collection involves placing a tube at the focal point of a parabolic reflector and passing a fluid through the tube.

The net effect of this arrangement may be approximated as one of creating a condition of uniform heating at the surface of the tube. That is, the resulting heat flux to the fluid $q''_s$ may be assumed to be a constant along the circumference and axis of the tube. Consider operation with a tube of diameter $D = 60$ mm on a sunny day for which $q''_s = 2000$ W/m$^2$.

1. If pressurized water enters the tube at $\dot{m} = 0.01$ kg/s and $T_{in} = 20^\circ$C, what tube length $L$ is required to obtain an exit temperature of $80^\circ$C?

2. What is the surface temperature at the outlet of the tube, where fully developed conditions may be assumed to exist?
SOLUTION

Known: Internal flow with uniform surface heat flux.

Find:
1. Length of tube $L$ to achieve required heating.
2. Surface temperature $T_s(L)$ at the outlet section, $x = L$.

Schematic:

Assumptions:
1. Steady-state conditions.
2. Incompressible liquid and negligible viscous dissipation.
3. Constant properties.
4. Fully developed conditions at tube outlet.

Properties: Table A.6, water ($\bar{T}_w = 323$ K): $c_p = 4181$ J/kg · K. Table A.6, water ($T_{m,o} = 353$ K): $k = 0.670$ W/m · K, $\mu = 352 \times 10^{-6}$ N · s/m², $Pr = 2.2$.

Analysis:

1. For constant surface heat flux, Equation 8.38 may be used with the energy balance, Equation 8.34, to obtain

$$A_s = \pi DL = -\frac{\dot{m}c_p(T_{m,o} - T_{m,i})}{q_s}$$

$$L = \frac{\dot{m}c_p}{\pi DL} (T_{m,o} - T_{m,i})$$

Hence

$$L = \frac{0.01 \text{ kg/s} \times 4181 \text{ J/kg} \cdot \text{K}}{\pi \times 0.060 \text{ m} \times 2000 \text{ W/m}^2} (80 - 20)\text{°C} = 6.65 \text{ m}$$

2. The surface temperature at the outlet may be obtained from Newton's law of cooling, Equation 8.27, where

$$T_s = \frac{q_s}{h} + T_{m,o}$$
To find the local convection coefficient at the tube outlet, the nature of the flow condition must first be established. From Equation 8.6,

$$Re_D = \frac{\frac{4m}{\pi D \mu}}{\mu} = \frac{4 \times 0.01 \text{ kg/s}}{\pi \times 0.060 \text{ m} \times 352 \times 10^{-6} \text{ N} \cdot \text{s/m}^2} = 603$$

Hence the flow is laminar. With the assumption of fully developed conditions, the appropriate heat transfer correlation is then

$$Nu_D = \frac{hD}{k} = 4.36$$

and

$$h = 4.36 \frac{k}{D} = 4.36 \frac{0.670 \text{ W/m} \cdot \text{K}}{0.06 \text{ m}} = 48.7 \text{ W/m}^2 \cdot \text{K}$$

The surface temperature at the tube outlet is then

$$T_{s.o} = \frac{2000 \text{ W/m}^2}{48.7 \text{ W/m}^2 \cdot \text{K}} + 80^\circ \text{C} = 121^\circ \text{C}$$

Comments: For the conditions given, \((x_0/D) = 0.05Re_DPr = 66.3\), while \(L/D = 110\). Hence the assumption of fully developed conditions is justified. Note, however, that with \(T_{s.o} > 100^\circ \text{C}\), boiling may occur at the tube surface.

**Example 8.5**

In the human body, blood flows from the heart in a series of branching blood vessels having successively smaller diameters. The capillaries are the smallest blood vessels. In developing the bioheat equation (Section 3.7), Pennes assumed that blood enters the capillaries at the arterial temperature and exits at the temperature of the surrounding tissue. This problem tests that assumption [5,6]. The diameters and average blood velocities for three different types of blood vessels are given in the table below. For each of these blood vessels, estimate the length required for the mean blood temperature to closely approach the tissue temperature, specifically, to satisfy \((T_t - T_{n.a})/(T_t - T_{m.t}) = 0.05\). The heat transfer between the vessel wall and surrounding tissue can be approximated by an effective heat transfer coefficient, \(h_t = k_t/D\), where \(k_t = 0.5 \text{ W/m} \cdot \text{K}\).

<table>
<thead>
<tr>
<th>Vessel</th>
<th>Diameter, (D) (mm)</th>
<th>Blood Velocity, (u_m) (mm/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Large artery</td>
<td>3</td>
<td>130</td>
</tr>
<tr>
<td>Arteriole</td>
<td>0.02</td>
<td>3</td>
</tr>
<tr>
<td>Capillary</td>
<td>0.008</td>
<td>0.7</td>
</tr>
</tbody>
</table>

**Solution**

**Known:** Blood vessel diameter and average blood velocity. Tissue thermal conductivity and effective heat transfer coefficient.
Find: Length of blood vessel needed to satisfy $(T_i - T_{m,w})(T_i - T_{m,t}) = 0.05$.

Schematic:

Assumptions:
1. Steady-state conditions.
2. Constant properties.
3. Negligible blood vessel wall thermal resistance.
4. Thermal properties of blood can be approximated by those of water.
5. Blood is incompressible liquid with negligible viscous dissipation.
6. Tissue temperature is fixed.
7. Effects of pulsation of flow are negligible.

Properties: Table A.6, water $(\bar{T}_m = 310 \, \text{K}) \rho = \nu_f^{-1} = 993 \, \text{kg/m}^3, c_p = 4178 \, \text{J/kg} \cdot \text{K}, \mu = 695 \times 10^{-6} \, \text{N} \cdot \text{s/m}^2, \kappa = 0.628 \, \text{W/m} \cdot \text{K}, Pr = 4.62$.

Analysis: Since the tissue temperature is fixed and heat transfer between the blood vessel wall and the tissue can be represented by an effective heat transfer coefficient, Equation 8.45a is applicable, with the “free stream” temperature equal to the tissue temperature, $T_i$. This equation can be used to find the required length $L$, since $A_i = \pi DL$. However, we must first find $U$, which requires knowledge of the heat transfer coefficient for the blood flow, $h_b$. Taking the large artery as an example, the Reynolds number is given by

$$Re_D = \frac{\rho u_m D}{\mu} = \frac{993 \, \text{kg/m}^3 \times 130 \times 10^{-3} \, \text{m/s} \times 3 \times 10^{-3} \, \text{m}}{695 \times 10^{-6} \, \text{N} \cdot \text{s/m}^2} = 557$$

so the flow is laminar. Since the other blood vessels have smaller diameters and velocities, their flows will also be laminar. Because we don’t yet know the length of the vessel, we don’t know whether the flow becomes fully developed. However, we will begin by assuming fully developed conditions. Moreover, because the situation is neither one of constant surface temperature nor constant surface heat flux, we will estimate the required length by approximating the Nusselt number as $Nu = 4$, in which case $h_b = 4k_b/D$. Neglecting the thermal resistance of the vessel wall, for the large artery

$$\frac{1}{U} = \frac{1}{h_b} + \frac{1}{h_i} = \frac{D}{4k_b} + \frac{D}{k_i} = \frac{3 \times 10^{-3} \, \text{m}}{4 \times 0.628 \, \text{W/m} \cdot \text{K}} + \frac{3 \times 10^{-3} \, \text{m}}{0.5 \, \text{W/m} \cdot \text{K}} = 7.2 \times 10^{-3} \, \text{m}^2 \cdot \text{K/W}$$
or

\[ \bar{U} = 140 \text{ W/m}^2 \cdot \text{K} \]

The length can then be found by solving Equation 8.45a, with \( m = \rho \mu_n \pi D^2 / 4 \):

\[
L = \frac{\rho \mu_n D c_p}{4 \bar{U}} \ln \left( \frac{T_i - T_{\text{in}}}{T_i - T_{\text{st}}} \right)
= \frac{993 \text{ kg/m}^3 \times 130 \times 10^{-3} \text{ m/s} \times 3 \times 10^{-3} \text{ m} \times 4178 \text{ J/kg} \cdot \text{K}}{4 \times 140 \text{ W/m}^2 \cdot \text{K}} \ln(0.05)
= 8.7 \text{ m}
\]

We can now test the assumption that the flow is hydrodynamically and thermally fully developed, using Equations 8.3 and 8.23:

\[
x_{\text{fd,h}} = 0.05 \text{Re}_D D = 0.05 \times 557 \times 3 \times 10^{-3} \text{ m} = 0.08 \text{ m}
\]

\[
x_{\text{fd,l}} = x_{\text{fd,h}} P_t = 0.08 \text{ m} \times 4.62 = 0.4 \text{ m}
\]

The flow would indeed be fully developed well within the length of 8.7 m. The calculations can be repeated for the other two cases and are tabulated below.

<table>
<thead>
<tr>
<th>Vessel</th>
<th>( \text{Re}_D )</th>
<th>( \bar{U} ) (W/m²·K)</th>
<th>( L ) (m)</th>
<th>( x_{\text{fd,h}} ) (m)</th>
<th>( x_{\text{fd,l}} ) (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Large artery</td>
<td>557</td>
<td>140</td>
<td>8.7</td>
<td>0.08</td>
<td>0.4</td>
</tr>
<tr>
<td>Arteriole</td>
<td>0.086</td>
<td>21,000</td>
<td>8.9 \times 10^{-6}</td>
<td>9 \times 10^{-8}</td>
<td>4 \times 10^{-7}</td>
</tr>
<tr>
<td>Capillary</td>
<td>0.0080</td>
<td>52,000</td>
<td>3.3 \times 10^{-7}</td>
<td>3 \times 10^{-9}</td>
<td>1 \times 10^{-8}</td>
</tr>
</tbody>
</table>

**Comments:**

1. The blood temperature in the large artery approaches the tissue temperature very slowly. This is due to its relatively large diameter, which leads to a small overall heat transfer coefficient. Thus, the temperature in large arteries remains close to the inlet arterial blood temperature.

2. In arterioles, the blood temperature comes close to the tissue temperature within a length on the order of 10 μm. Since arterioles are on the order of millimeters in length, the blood temperature exiting them would be essentially the same as the tissue temperature. There would then be no further temperature drop in the capillaries, which are the next smallest vessel. Thus, it is in the arterioles and slightly larger vessels in which the blood temperature equilibrates to the tissue temperature, not in the capillaries as Pennes described. Despite this flaw, the bioheat equation has proved a useful tool in analyzing heat transfer in the human body.

3. The properties of blood are moderately close to those of water. The property which differs most is the viscosity, as blood is more viscous than water. However, that change would have no effect on the foregoing calculations. Since the Reynolds number would be even smaller, the flow would still be laminar and the heat transfer would be unaffected.

4. Individual blood cells have dimensions on the order of the capillary diameter. Thus, for the capillaries, an accurate model of blood flow would account for the individual cells surrounded by plasma.
8.4.2 The Entry Region

The energy equation for the entry region is more complicated than Equation 8.48 because there would be a radial advection term (since \( v \neq 0 \) in the entry region). In addition, both velocity and temperature now depend on \( x \), as well as \( r \), and the axial temperature gradient \( \delta T/\delta x \) may no longer be simplified through Equations 8.32 or 8.33. However, two different entry length solutions have been obtained. The simplest solution is for the thermal entry length problem, and it is based on assuming that thermal conditions develop in the presence of a fully developed velocity profile. Such a situation would exist if the location at which heat transfer begins were preceded by an unheated starting length. It could also be assumed to a reasonable approximation for large Prandtl number fluids, such as oils. Even in the absence of an unheated starting length, velocity boundary layer development would occur far more rapidly than thermal boundary layer development for large Prandtl number fluids, and a thermal entry length approximation could be made. In contrast, the combined (thermal and velocity) entry length problem corresponds to the case for which the temperature and velocity profiles develop simultaneously.

Solutions have been obtained for both entry length conditions [2], and selected results are shown in Figure 8.10. As evident in Figure 8.10a, local Nusselt numbers, \( Nu_{x} \), are, in principle, infinite at \( x = 0 \) and decay to their asymptotic (fully developed) values with increasing \( x \). When plotted against the dimensionless parameter

![Figure 8.10](image-url)

**Figure 8.10** Results obtained from entry length solutions for laminar flow in a circular tube: (a) local Nusselt numbers and (b) average Nusselt numbers [2].
\[
\frac{x}{\bar{u}_r D^2} = \frac{x}{(D \xi R_E \Pr)} = \frac{0.0668(D/L)Re_D \Pr}{1 + 0.04[(D/L)Re_D \Pr]^{2/3}}
\]

(8.56)

where \( \bar{N}_\text{ud} = \bar{h} D / k \). Because this result is for the thermal entry length problem, it is applicable to all situations where the velocity profile is already fully developed. For the combined entry length, a suitable correlation for use at moderate Prandtl numbers, due to Sieder and Tate [9], is of the form

\[
\bar{N}_\text{ud} = 1.86 \left( \frac{Re_D \Pr}{L/D} \right)^{0.13} \left( \frac{\mu}{\mu_\text{s}} \right)^{0.14}
\]

(8.57)

Equation 8.57 is recommended for use when \( 0.60 \leq \Pr \leq 5 \), provided \( \bar{N}_\text{ud} \geq 3.66 \) [2, 10]. If \( \bar{N}_\text{ud} \) falls below this value, it is reasonable to use \( \bar{N}_\text{ud} = 3.66 \), since fully developed conditions then encompass much of the tube. For larger Prandtl numbers (\( \Pr \geq 5 \)), hydrodynamic conditions develop much faster than thermal conditions, and Equation 8.56 is recommended instead of Equation 8.57 [2]. Equations 8.56 and 8.57, along with numerical predictions of \( \bar{N}_\text{ud} \) versus \( \sqrt{x/(D \xi R_E \Pr)} \) and \( \Pr \), are shown in Figure 8.10b. All properties appearing in Equations 8.56 and 8.57, except \( \mu_\text{s} \), should be evaluated at the average value of the mean temperature, \( T_m = (T_{\text{in}} + T_{\text{out}}) / 2 \).

The subject of laminar flow in ducts has been studied extensively, and numerous results are available for a variety of duct cross sections and surface conditions. These results have been compiled in a monograph by Shah and London [11] and in an updated review by Shah and Bhatti [12].
8.5
Convection Correlations: Turbulent Flow in Circular Tubes

Since the analysis of turbulent flow conditions is a good deal more involved, greater emphasis is placed on determining empirical correlations. A classical expression for computing the local Nusselt number for fully developed (hydrodynamically and thermally) turbulent flow in a smooth circular tube is due to Colburn [13] and may be obtained from the Chilton-Colburn analogy. Substituting Equation 6.38 into Equation 8.18, the analogy is of the form

\[
\frac{C_f}{2} = \frac{f}{8} = St \cdot Pr^{0.33} = \frac{Nu_D}{Re_D \cdot Pr^{0.33}}
\]

Substituting for the friction factor from Equation 8.21, the Colburn equation is then

\[
Nu_D = 0.023 Re_D^{0.6} \cdot Pr^{0.33}
\]

(8.59)

The Dittus-Boelter equation [14] is a slightly different and preferred version of the above result and is of the form\(^1\)

\[
Nu_D = 0.023 Re_D^{0.85} \cdot Pr^n
\]

(8.60)

where \(n = 0.4\) for heating \((T_i > T_w)\) and \(0.3\) for cooling \((T_i < T_w)\). These equations have been confirmed experimentally for the range of conditions

\[
\begin{align*}
0.6 \leq Pr &\leq 160 \\
Re_D &\geq 10,000 \\
\frac{L}{D} &\geq 10
\end{align*}
\]

The equations may be used for small to moderate temperature differences, \(T_i - T_w\) with all properties evaluated at \(T_w\). For flows characterized by large property variations, the following equation, due to Sieder and Tate [9], is recommended:

\[
Nu_D = 0.027 Re_D^{0.62} \cdot Pr^{0.13} \left( \frac{\mu_i}{\mu_L} \right)^{0.14}
\]

(8.61)

\[
\begin{align*}
0.7 \leq Pr &\leq 16,700 \\
Re_D &\geq 10,000 \\
\frac{L}{D} &\geq 10
\end{align*}
\]

\(^1\)Although it has become common practice to refer to Equation 8.60 as the Dittus-Boelter equation, the original Dittus-Boelter equations are actually of the form

\[
\begin{align*}
Nu_D = 0.0243 Re_D^{0.6} \cdot Pr^{0.4} & \quad (\text{Heating}) \\
Nu_D = 0.0265 Re_D^{0.6} \cdot Pr^{0.3} & \quad (\text{Cooling})
\end{align*}
\]

The historical origins of Equation 8.60 are discussed by Winterton [14].
where all properties except $\mu_r$ are evaluated at $T_w$. To a good approximation, the foregoing correlations may be applied for both the uniform surface temperature and heat flux conditions.

Although Equations 8.60 and 8.61 are easily applied and are certainly satisfactory for the purposes of this text, errors as large as 25% may result from their use. Such errors may be reduced to less than 10% through the use of more recent, but generally more complex, correlations [4, 15]. One correlation, valid over a large Reynolds number range including the transition region, is provided by Gnielinski [16]:

$$
N_u = \frac{(f/8)(Re_r - 1000)Pr}{1 + 12.7(f/8)^{1/2}(Pr^{2/3} - 1)}
$$

(8.62)

where the friction factor may be obtained from the Moody diagram, or, for smooth tubes, from Equation 8.21. The correlation is valid for $0.5 \leq Pr \leq 2000$ and $3000 \leq Re_r \leq 5 \times 10^6$. In using Equation 8.62, which applies for both uniform surface heat flux and temperature, properties should be evaluated at $T_m$. If temperature differences are large, additional consideration must be given to variable-property effects and available options are reviewed by Kakac [17].

We note that, unless specifically developed for the transition region ($2300 < Re_r < 10^4$), caution should be exercised when applying a turbulent flow correlation for $Re_r < 10^4$. If the correlation was developed for fully turbulent conditions ($Re_r > 10^4$), it may be used as a first approximation at smaller Reynolds numbers, with the understanding that the convection coefficient will be overpredicted. If a higher level of accuracy is desired, the Gnielinski correlation, Equation 8.62, may be used. A comprehensive discussion of heat transfer in the transition region is provided by Ghajar and Tam [18].

We also note that Equations 8.59 through 8.62 pertain to smooth tubes. For turbulent flow, the heat transfer coefficient increases with wall roughness, and, as a first approximation, it may be computed by using Equation 8.62 with friction factors obtained from the Moody diagram, Figure 8.3. However, although the general trend is one of increasing $h$ with increasing $f$, the increase in $f$ is proportionately larger, and when $f$ is approximately four times larger than the corresponding value for a smooth surface, $h$ no longer changes with additional increases in $f$ [19]. Procedures for estimating the effect of wall roughness on convection heat transfer in fully developed turbulent flow are discussed by Bhatti and Shah [15].

Since entry lengths for turbulent flow are typically short, $10 \leq (x_d/D) \leq 60$, it is often reasonable to assume that the average Nusselt number for the entire tube is equal to the value associated with the fully developed region, $\bar{N}_u = N_u_{1,6}$. However, for short tubes $\bar{N}_u_{1,6}$ will exceed $N_u_{1,6}$ and may be calculated from an expression of the form

$$
\frac{\bar{N}_u}{N_u_{1,6}} = 1 + \frac{C}{(x/d)^m}
$$

(8.63)

where $C$ and $m$ depend on the nature of the inlet (e.g., sharp-edged or nozzle) and entry region (thermal or combined), as well as on the Prandtl and Reynolds numbers [2, 15, 20]. Typically, errors of less than 15% are associated with assuming
\[ \overline{N_u_D} = N_u_{D,M} \text{ for } (L/D) > 60. \] When determining \( \overline{N_u_D} \), all fluid properties should be evaluated at the arithmetic average of the mean temperature, \( T_m = (T_{m,i} + T_{m,o})/2 \).

Finally, we note that the foregoing correlations do not apply to liquid metals \((3 \times 10^{-3} \leq Pr \leq 5 \times 10^{-7})\). For fully developed turbulent flow in smooth circular tubes with constant surface heat flux, Skupinski et al. [21] recommend a correlation of the form

\[
N_u_D = 4.82 + 0.0185 P_e_{D}^{0.037} \quad q''_{i} = \text{constant} \quad (8.64)
\]

\[
\begin{bmatrix}
3.6 \times 10^{3} \leq Re_{D} \leq 9.05 \times 10^{5} \\
10^{3} \leq Pe_{D} \leq 10^{4}
\end{bmatrix}
\]

Similarly, for constant surface temperature Seban and Shimazaki [22] recommend the following correlation for \( Pe_{D} \geq 100 \):

\[
N_u_D = 5.0 + 0.025 P_e_{D}^{0.8} \quad T_s = \text{constant} \quad (8.65)
\]

Extensive data and additional correlations are available in the literature [23].

**Example 8.6**

Hot air flows with a mass rate of \( \dot{m} = 0.050 \text{ kg/s} \) through an uninsulated sheet metal duct of diameter \( D = 0.15 \text{ m} \), which is in the crawlspace of a house. The hot air enters at \( 103^\circ \text{C} \) and, after a distance of \( L = 5 \text{ m} \), cools to \( 77^\circ \text{C} \). The heat transfer coefficient between the duct outer surface and the ambient air at \( T_a = 0^\circ \text{C} \) is known to be \( h_a = 6 \text{ W/m}^2 \cdot K \).

1. Calculate the heat loss (W) from the duct over the length \( L \).
2. Determine the heat flux and the duct surface temperature at \( x = L \).

**Solution**

**Known:** Hot air flowing in a duct.

**Find:**
1. Heat loss from the duct over the length \( L \), \( q \) (W).
2. Heat flux and surface temperature at \( x = L \).

**Schematic:**

\[ \text{Hot air} \quad \dot{m} = 0.05 \text{ kg/s} \quad T_{m,i} = 103^\circ \text{C} \]

**Assumptions:**
1. Steady-state conditions.
2. Constant properties.
3. Ideal gas behavior.
4. Negligible viscous dissipation and negligible pressure variations.
5. Negligible duct wall thermal resistance.
6. Uniform convection coefficient at outer surface of duct.

Properties: Table A.4, air \((T_p = 363\, \text{K})\): \(c_p = 1010\, \text{J/kg} \cdot \text{K}\). Table A.4, air \((T_{m,l} = 350\, \text{K})\): \(k = 0.030\, \text{W/m} \cdot \text{K}, \mu = 208.2 \times 10^{-7}\, \text{N} \cdot \text{s/m}^2, Pr = 0.70\).

Analysis:
1. From the energy balance for the entire tube, Equation 8.34,
   \[ q = \dot{m}c_p(T_{m,l} - T_m) \]
   \[ q = 0.05 \, \text{kg/s} \times 1010\, \text{J/kg} \cdot \text{K} \cdot (77 - 103)\, ^\circ\text{C} = -1313\, \text{W} \]

2. An expression for the heat flux at \(x = L\) may be inferred from the resistance network

   \[ q''_L(L) \rightarrow T_{m,l} \rightarrow T_s(L) \rightarrow T_w \]

   where \(h_s(L)\) is the inside convection heat transfer coefficient at \(x = L\). Hence

   \[ q''_L(L) = \frac{T_{m,l} - T_s}{1/h_s(L) + (1/h_w)} \]

   The inside convection coefficient may be obtained from knowledge of the Reynolds number. From Equation 8.6

   \[ Re_p = \frac{\dot{V}L}{\pi D\mu} = \frac{4 \times 0.05\, \text{kg/s}}{\pi \times 0.15\, \text{m} \times 208.2 \times 10^{-7}\, \text{N} \cdot \text{s/m}^2} = 20,384 \]

   Hence the flow is turbulent. Moreover, with \((L/D) = (50/1.15) = 33.3\), it is reasonable to assume fully developed conditions at \(x = L\). Hence from Equation 8.60, with \(n = 0.3\),

   \[ Nu_D = \frac{h_s(L)D}{k} = 0.023\, Re_p^{0.5} Pr^{0.3} = 0.023(20,384)^{0.5}(0.70)^{0.3} = 57.9 \]

   \[ h_s(L) = Nu_D \frac{k}{D} = 57.9 \frac{0.030\, \text{W/m} \cdot \text{K}}{0.15\, \text{m}} = 11.6\, \text{W/m}^2 \cdot \text{K} \]

   Hence

   \[ q''_L(L) = \frac{(77 - 0)^\circ\text{C}}{[(1/11.6) + (1/6.0)]\, \text{m}^2 \cdot \text{W/K}} = 304.5\, \text{W/m}^2 \]

   Referring back to the network, it also follows that

   \[ q''_L(L) = \frac{T_{m,l} - T_s}{1/h_s(L)} \]

   in which case

   \[ T_{s,L} = T_{m,l} - \frac{q''_L(L)}{h_s(L)} = 77\, ^\circ\text{C} - \frac{304.5\, \text{W/m}^2}{11.6\, \text{W/m}^2 \cdot \text{K}} = 50.7\, ^\circ\text{C} \]