PROBLEM 4.2

KNOWN: Two-dimensional rectangular plate subjected to prescribed uniform temperature boundary conditions.

FIND: Temperature at the mid-point using the exact solution considering the first five non-zero terms; assess error resulting from using only first three terms. Plot the temperature distributions T(x,0.5) and T(1,y).

SCHEMATIC:



ASSUMPTIONS: (1) Two-dimensional, steady-state conduction, (2) Constant properties.

ANALYSIS: From Section 4.2, the temperature distribution is

$$\theta(\mathbf{x}, \mathbf{y}) = \frac{\mathbf{T} - \mathbf{T}_{1}}{\mathbf{T}_{2} - \mathbf{T}_{1}} = \frac{2}{\pi} \sum_{n=1}^{\theta} \frac{(-1)^{n+1} + 1}{n} \sin\left(\frac{n\pi \mathbf{x}}{L}\right) \cdot \frac{\sinh\left(n\pi \mathbf{y}/L\right)}{\sinh\left(n\pi \mathbf{W}/L\right)}.$$
 (1,4.19)

Considering now the point (x,y) = (1.0,0.5) and recognizing x/L = 1/2, y/L = 1/4 and W/L = 1/2,

$$\theta(1,0.5) = \frac{T - T_1}{T_2 - T_1} = \frac{2}{\pi} \sum_{n=1}^{\theta} \frac{(-1)^{n+1} + 1}{n} \sin\left(\frac{n\pi}{2}\right) \cdot \frac{\sinh(n\pi/4)}{\sinh(n\pi/2)}$$

When n is even (2, 4, 6 ...), the corresponding term is zero; hence we need only consider n = 1, 3, 5, 7 and 9 as the first five non-zero terms.

$$\theta(1,0.5) = \frac{2}{\pi} \left\{ 2\sin\left(\frac{\pi}{2}\right) \frac{\sinh\left(\pi/4\right)}{\sinh\left(\pi/2\right)} + \frac{2}{3}\sin\left(\frac{3\pi}{2}\right) \frac{\sinh\left(3\pi/4\right)}{\sinh\left(3\pi/2\right)} + \frac{2}{5}\sin\left(\frac{5\pi}{2}\right) \frac{\sinh\left(5\pi/4\right)}{\sinh\left(5\pi/2\right)} + \frac{2}{7}\sin\left(\frac{7\pi}{2}\right) \frac{\sinh\left(7\pi/4\right)}{\sinh\left(7\pi/2\right)} + \frac{2}{9}\sin\left(\frac{9\pi}{2}\right) \frac{\sinh\left(9\pi/4\right)}{\sinh\left(9\pi/2\right)} \right\}$$
$$\theta(1,0.5) = \frac{2}{\pi} \left[0.755 - 0.063 + 0.008 - 0.001 + 0.000 \right] = 0.445$$
(2)
$$T(1,0.5) = \theta(1,0.5)(T_2 - T_1) + T_1 = 0.445(150 - 50) + 50 = 94.5^{\circ}C.$$

If only the first three terms of the series, Eq. (2), are considered, the result will be $\theta(1,0.5) = 0.46$; that is, there is less than a 0.2% effect.

Using Eq. (1), and writing out the first five terms of the series, expressions for $\theta(x,0.5)$ or T(x,0.5) and $\theta(1,y)$ or T(1,y) were keyboarded into the IHT workspace and evaluated for sweeps over the x or y variable. Note that for T(1,y), that as $y \rightarrow 1$, the upper boundary, T(1,1) is greater than 150°C. Upon examination of the magnitudes of terms, it becomes evident that more than 5 terms are required to provide an accurate solution.

