PROBLEM 2.13

KNOWN: A rod of constant thermal conductivity $k$ and variable cross-sectional area $A_x(x) = A_0 e^{ax}$ where $A_0$ and $a$ are constants.

FIND: (a) Expression for the conduction heat rate, $q_x(x)$; use this expression to determine the temperature distribution, $T(x)$; and sketch of the temperature distribution, (b) Considering the presence of volumetric heat generation rate, $\dot{q} = \dot{q}_o \exp(-ax)$, obtain an expression for $q_x(x)$ when the left face, $x = 0$, is well insulated.

SCHEMATIC:

ASSUMPTIONS: (1) One-dimensional conduction in the rod, (2) Constant properties, (3) Steady-state conditions.

ANALYSIS: Perform an energy balance on the control volume, $A(x)\,dx$,

$$E_{\text{in}} - E_{\text{out}} + E_{E} = 0$$

$$q_{x} - q_{x} + \dot{q} \cdot A(x) \, dx = 0$$

The conduction heat rate terms can be expressed as a Taylor series and substituting expressions for $\dot{q}$ and $A(x)$,

$$- \frac{d}{dx} (q_{x}) + \dot{q}_o \exp(-ax) \cdot A_0 \exp(ax) = 0 \quad \text{(1)}$$

$$q_{x} = -k \cdot A(x) \frac{dT}{dx} \quad \text{(2)}$$

(a) With no internal generation, $\dot{q}_o = 0$, and from Eq. (1) find

$$- \frac{d}{dx} (q_{x}) = 0$$

indicating that the heat rate is constant with $x$. By combining Eqs. (1) and (2)

$$- \frac{d}{dx} \left( -k \cdot A(x) \frac{dT}{dx} \right) = 0 \quad \text{or} \quad A(x) \frac{dT}{dx} = C_1$$

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PROBLEM 2.13 (Cont.)

That is, the product of the cross-sectional area and the temperature gradient is a constant, independent of x. Hence, with T(0) > T(L), the temperature distribution is exponential, and as shown in the sketch above. Separating variables and integrating Eq. (3), the general form for the temperature distribution can be determined,

\[ A_0 \exp(ax) \frac{dT}{dx} = C_1 \]

\[ dT = C_1 A_0^{-1} \exp(-ax) dx \]

\[ T(x) = -C_1 A_0 a \exp(-ax) + C_2 \]

We could use the two temperature boundary conditions, \( T_o = T(0) \) and \( T_L = T(L) \), to evaluate \( C_1 \) and \( C_2 \) and, hence, obtain the temperature distribution in terms of \( T_o \) and \( T_L \).

(b) With the internal generation, from Eq. (1),

\[ -\frac{d}{dx}(q_x) + \dot{q}_o A_o = 0 \quad \text{or} \quad q_x = \dot{q}_o A_o x \]

That is, the heat rate increases linearly with x.

**COMMENTS:** In part (b), you could determine the temperature distribution using Fourier’s law and knowledge of the heat rate dependence upon the x-coordinate. Give it a try!