MAE 101 A
SOLUTIONS FOR HOMEWORK #6

4.1 An idealized velocity field is given by the formula

$$\mathbf{V} = 4tx\mathbf{i} - 2t^2y\mathbf{j} + 4xz\mathbf{k}$$

Is this flow field steady or unsteady? Is it two- or three-dimensional? At the point $(x, y, z) = (-1, +1, 0)$, compute (a) the acceleration vector and (b) any unit vector normal to the acceleration.

Solution: (a) The flow is unsteady because time t appears explicitly in the components. (b) The flow is three-dimensional because all three velocity components are nonzero. (c) Evaluate, by laborious differentiation, the acceleration vector at (x, y, z) = (−1, +1, 0).

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = 4x + 4tx(4t) - 2t^2y(0) + 4xz(0) = 4x + 16t^2x$$

$$\frac{dv}{dt} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -4ty + 4tx(0) - 2t^2y(-2t^2) + 4xz(0) = -4ty + 4t^4y$$

$$\frac{dw}{dt} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = 0 + 4tx(4z) - 2t^2y(0) + 4xz(4x) = 16txz + 16x^2z$$

$$\text{or:} \quad \frac{d\mathbf{V}}{dt} = (4x + 16t^2x)\mathbf{i} + (-4ty + 4t^4y)\mathbf{j} + (16txz + 16x^2z)\mathbf{k}$$

at (x, y, z) = (−1, +1, 0), we obtain $\frac{d\mathbf{V}}{dt} = -4(1 + 4t^2)\mathbf{i} - 4t(1 - t^3)\mathbf{j} + 0\mathbf{k}$ *Ans.* (c)

(d) At (−1, +1, 0) there are many unit vectors normal to $d\mathbf{V}/dt$. One obvious one is **k**. *Ans.*

P4.4 A simple flow model for a two-dimensional converging nozzle is the distribution

$$u = U_o\left(1 + \frac{x}{L}\right) \qquad v = -U_o\frac{y}{L} \qquad w = 0$$

(*a*) Sketch a few streamlines in the region $0 < x/L < 1$ and $0 < y/L < 1$, using the method of Section 1.11. (*b*) Find expressions for the horizontal and vertical accelerations.

(c) Where is the largest resultant acceleration and its numerical value?

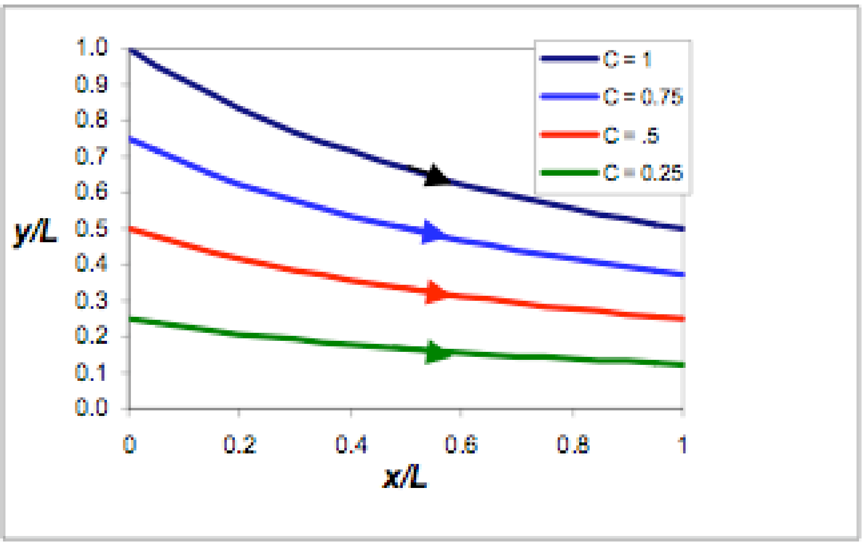
Solution: The streamlines are in the x-y plane and are found from the velocities:

$$\frac{dx}{u} = \frac{dy}{v} \quad \text{or integrate:} \quad \int \frac{dx}{U_o(1+x/L)} = -\int \frac{dy}{U_o y/L} \quad \text{Cancel } U_o$$

$$L\ln(1+x/L) = -L\ln(y/L) + const, \quad or: \ \ln[(y/L)(1+x/L)] = constant$$

$$\text{Finally the streamlines:} \quad \mathbf{\frac{y}{L} = \frac{C}{1+x/L}} \qquad \textit{Ans.(a)}$$

These may be plotted for various values of the dimensionless constant C, as shown:



The streamlines converge and the velocity increases to the right. *Ans.(a)*

(b) The accelerations are calculated from Eq. (4.2):

$$a_x = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = [U_o(1+x/L)](U_o/L) + 0 = \frac{U_o^2}{L}(1+\frac{x}{L})$$

$$a_y = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = 0 + (-U_o y/L)(-U_o/L) = \frac{U_o^2}{L}\frac{y}{L} \qquad \textit{Ans.}(b)$$

(*c*) Find the resultant of a_x and a_y from *Ans.*(*b*) above and introduce *y/L* from *Ans.*(*a*):

$$a = \sqrt{a_x^2 + a_y^2} = \frac{U_o^2}{L}\sqrt{(1+\frac{x}{L})^2 + (\frac{y}{L})^2} = \frac{U_o^2}{L}\sqrt{(1+\frac{x}{L})^2 + (\frac{C}{1+x/L})^2} \quad \textit{Ans.}(c)$$

$$0 \le x/L \le 1: \quad a_{max} = a_{x=L} = \frac{U_o^2}{L}\sqrt{4 + C^2/4} \qquad \textit{Ans.}(c)$$

We observe that the resultant acceleration increases with *x* and is greatest at $x = L$:

P4.16 Consider the plane polar coordinate velocity distribution

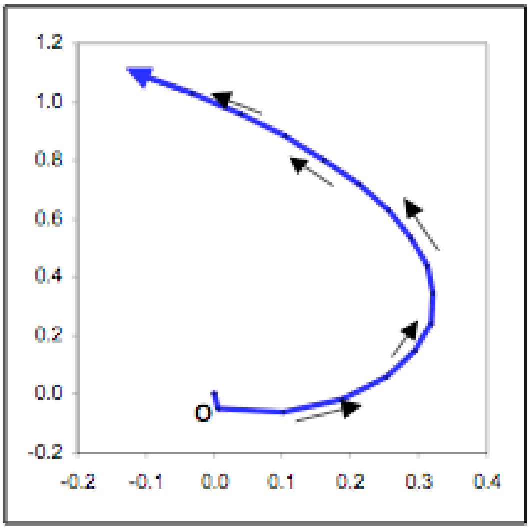
$$v_r = \frac{C}{r} \qquad v_\theta = \frac{K}{r} \qquad v_z = 0$$

where *C* and *K* are constants. (*a*) Determine if the incompressible equation of continuity is satisfied. (*b*) By sketching some velocity vector directions, plot a single streamline for $C = K$. What might this flow field simulate?

Solution: (*a*) Evaluate the incompressible continuity equation (4.12*b*) in polar coordinates:

$$\frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(v_\theta) = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{C}{r}) + \frac{1}{r}\frac{\partial}{\partial \theta}(\frac{K}{r}) = 0 + 0 = 0 \qquad \textit{Ans.}(a)$$

Incompressible continuity is indeed satisfied. (*b*) For $C = K$, we can plot a representative streamline by putting in some velocity vectors and sketching a line parallel to them:



The streamlines are logarithmic spirals moving out from the origin. [They have axisymmetry about O.] This simple distribution is often used to simulate a swirling flow such as a tornado.

4.19 An incompressible flow field has the cylindrical velocity components $\upsilon_\theta = Cr$, $\upsilon_z = K(R^2 - r^2)$, $\upsilon_r = 0$, where C and K are constants and $r \le R$, $z \le L$. Does this flow satisfy continuity? What might it represent physically?

Solution: We check the incompressible continuity relation in cylindrical coordinates:

$$\frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0 = 0 + 0 + 0 \quad \textbf{satisfied identically} \quad \textit{Ans.}$$

This flow also satisfies (cylindrical) momentum and could represent laminar flow inside a tube of radius R whose outer wall (r = R) is rotating at uniform angular velocity.

4.26 Curvilinear, or streamline, coordinates are defined in Fig. P4.26, where n is normal to the streamline in the plane of the radius of curvature R. Show that Euler's frictionless momentum equation (4.36) in streamline coordinates becomes

$$-V\frac{\partial \theta}{\partial t} - \frac{V^2}{R} = -\frac{1}{\rho}\frac{\partial p}{\partial n} + g_n \qquad (2)$$

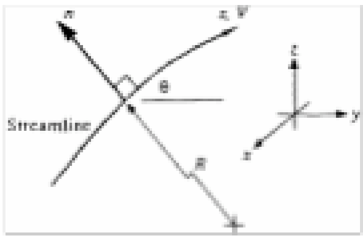


Fig. P4.26

Further show that the integral of Eq. (1) with respect to *s* is none other than our old friend Bernoulli's equation (3.76).

Solution: This is a laborious derivation, really, **the problem is only meant to *acquaint* the student with streamline coordinates.** The second part is not too hard, though. Multiply the streamwise momentum equation by *ds* and integrate:

$$\frac{\partial V}{\partial t}ds + V\,dV = -\frac{dp}{\rho_2} + g_s\,ds = -\frac{dp}{\rho} - g\sin\theta\,ds = -\frac{dp}{\rho} - g\,dz$$

Integrate from 1 to 2: $\displaystyle\int_1^2 \frac{\partial \mathbf{V}}{\partial t}\mathbf{ds} + \frac{\mathbf{V_2^2 - V_1^2}}{\mathbf{2}} + \int_1^2 \frac{\mathbf{dp}}{\rho} + \mathbf{g(z_2 - z_1) = 0}$ (Bernoulli) *Ans.*

4.27 A frictionless, incompressible steady-flow field is given by

$$\mathbf{V} = 2xy\mathbf{i} - y^2\mathbf{j}$$

in arbitrary units. Let the density be ρ_0 = constant and neglect gravity. Find an expression for the pressure gradient in the *x* direction.

Solution: For this (gravity-free) velocity, the momentum equation is

$$\rho\left(u\frac{\partial \mathbf{V}}{\partial x} + v\frac{\partial \mathbf{V}}{\partial y}\right) = -\nabla p, \quad \text{or:} \quad \rho_o[(2xy)(2y\mathbf{i}) + (-y^2)(2x\mathbf{i} - 2y\mathbf{j})] = -\nabla p$$

$$\text{Solve for} \quad \nabla p = -\rho_o(2xy^2\mathbf{i} + 2y^3\mathbf{j}), \quad \text{or:} \quad \frac{\partial \mathbf{p}}{\partial \mathbf{x}} = -\rho_0\mathbf{2xy^2} \quad \textit{Ans.}$$

4.28 If *z* is "up," what are the conditions on constants *a* and *b* for which the velocity field *u* = *ay*, *υ* = *bx*, *w* = 0 is an exact solution to the continuity and Navier-Stokes equations for incompressible flow?

Solution: First examine the continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \stackrel{?}{=} 0 = \frac{\partial}{\partial x}(ay) + \frac{\partial}{\partial y}(bx) + \frac{\partial}{\partial z}(0) = 0 + 0 + 0 \quad \text{for all } a \text{ and } b$$

Given gx = gy = 0 and w = 0, we need only examine *x*- and *y*-momentum:

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \rho[(ay)(0) + (bx)(a)] = -\frac{\partial p}{\partial x} + \mu(0+0)$$

$$\rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) = \rho[(ay)(b) + (bx)(0)] = -\frac{\partial p}{\partial y} + \mu(0+0)$$

Solve for $\frac{\partial p}{\partial x} = -\rho abx$ and $\frac{\partial p}{\partial y} = -\rho aby$, or: $\mathbf{p = -\frac{\rho}{2}ab(x^2 + y^2) + const}$ *Ans.*

The given velocity field, u = ay and v = bx, is an exact solution independent of a or b. It is not, however, an "irrotational" flow.

4.29 Consider a steady, two-dimensional, incompressible flow of a newtonian fluid with the velocity field $u = -2xy$, $v = y^2 - x^2$, and $w = 0$. (a) Does this flow satisfy conservation of mass? (b) Find the pressure field $p(x, y)$ if the pressure at point (x = 0, y = 0) is equal to p_a.

Solution: Evaluate and check the incompressible continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 = -2y + 2y + 0 \equiv 0 \quad \textbf{Yes!} \quad \textit{Ans. (a)}$$

(b) Find the pressure gradients from the Navier-Stokes *x*- and *y*-relations:

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right), \quad \textit{or:}$$

$$\rho[-2xy(-2y) + (y^2 - x^2)(-2x)] = -\frac{\partial p}{\partial x} + \mu(0+0+0), \quad \textit{or:} \quad \frac{\partial p}{\partial x} = -2\rho(xy^2 + x^3)$$

and, similarly for the *y*-momentum relation,

$$\rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right), \quad \textit{or:}$$

$$\rho[-2xy(-2x) + (y^2 - x^2)(2y)] = -\frac{\partial p}{\partial y} + \mu(-2+2+0), \quad \textit{or:} \quad \frac{\partial p}{\partial y} = -2\rho(x^2 y + y^3)$$

The two gradients $\partial p/\partial x$ and $\partial p/\partial y$ may be integrated to find p(x, y):

$$p = \int \frac{\partial p}{\partial x} dx\Big|_{y=Const} = -2\rho\left(\frac{x^2y^2}{2} + \frac{x^4}{4}\right) + f(y), \quad \textit{then differentiate}$$

$$\frac{\partial p}{\partial y} = -2\rho(x^2 y) + \frac{df}{dy} = -2\rho(x^2y + y^3), \quad \textit{whence} \quad \frac{df}{dy} = -2\rho y^3, \quad \textit{or:} \quad f(y) = -\frac{\rho}{2}y^4 + C$$

$$\textit{Thus:} \quad p = -\frac{\rho}{2}(2x^2y^2 + x^4 + y^4) + C = p_a \quad \textit{at } (x,y) = (0,0), \quad \textit{or:} \quad \mathbf{C = p_a}$$

Finally, the pressure field for this flow is given by

$$\mathbf{p = p_a - \frac{1}{2}\rho(2x^2y^2 + x^4 + y^4)} \quad \textit{Ans. (b)}$$

P4.30 For the velocity distribution of Prob. P4.4, determine if (*a*) the equation of continuity and (*b*) the Navier-Stokes equation are satisfied. (*c*) If the latter is true, find the pressure distribution $p(x,y)$ when the pressure at the origin equals p_o. Neglect gravity.

Solution: Recall that we were given $u = U_o(1+x/L)$ and $v = -U_o\,y/L$. (*a*) Test continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial}{\partial x}[U_o(1+\frac{x}{L})] + \frac{\partial}{\partial y}(-U_o\frac{y}{L}) = \frac{U_o}{L} - \frac{U_o}{L} = 0 \quad \text{OK, satisfied.} \quad \textit{Ans.(a)}$$

(*b*) Now substitute these velocities into the *x*- and *y*- Navier-Stokes:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U_o(1+\frac{x}{L})\frac{U_o}{L} + (-U_o\frac{y}{L})(0) = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2 u = -\frac{1}{\rho}\frac{\partial p}{\partial x} + 0$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = U_o(1+\frac{x}{L})(0) + (-U_o\frac{y}{L})(-\frac{U_o}{L}) = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\nabla^2 v = -\frac{1}{\rho}\frac{\partial p}{\partial y} + 0$$

Solve for the two pressure gradients and cross-differentiate to see if they agree:

$$\frac{\partial p}{\partial x} = -\frac{\rho U_o^2}{L}(1+\frac{x}{L}) \qquad \frac{\partial p}{\partial y} = -\frac{\rho U_o^2}{L}\frac{y}{L} \qquad \text{Check } \frac{\partial^2 p}{\partial x \partial y} = 0 \text{ for both}$$

Thus, before finding $p(x,y)$, we know this is **an exact solution to Navier-Stokes**. *Ans.(b)*

(c) Integrate the two pressure gradients to find the pressure distribution:

$$p = \int\frac{\partial p}{\partial x}dx = -\frac{\rho U_o^2}{L}(x+\frac{x^2}{2L}) + f(y)\,; \text{ Then } \frac{\partial p}{\partial y} = \frac{df}{dy},\ f = -\frac{\rho U_o^2}{L}\frac{y^2}{2L} + const$$

$$p = -\frac{\rho U_o^2}{L}(x+\frac{x^2}{2L}+\frac{y^2}{2L}) + p_o \qquad \textit{Ans.(c)}$$

This is the same as **Bernoulli's equation**, but that is a bit hard to see.

4.36 A constant-thickness film of viscous liquid flows in laminar motion down a plate inclined at angle θ, as in Fig. P4.36. The velocity profile is

$$u = Cy(2h - y) \quad v = w = 0$$

Find the constant C in terms of the specific weight and viscosity and the angle θ. Find the volume flux Q per unit width in terms of these parameters.

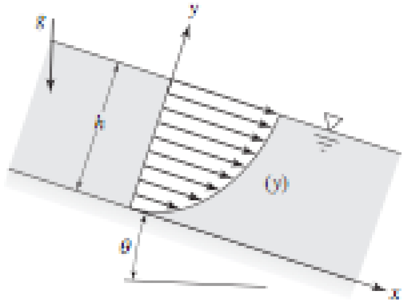


Fig. P4.36

Solution: There is atmospheric pressure all along the surface at y = h, hence $\partial p/\partial x =$
The x-momentum equation can easily be evaluated from the known velocity profile:

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \rho g_x + \mu\nabla^2 u, \quad \text{or:} \quad 0 = 0 + \rho g \sin\theta + \mu(-2C)$$

$$\text{Solve for} \quad \mathbf{C} = \frac{\boldsymbol{\rho g \sin\theta}}{\boldsymbol{2\mu}} \quad \textit{Ans. (a)}$$

The flow rate per unit width is found by integrating the velocity profile and using C:

$$Q = \int_0^h u\,dy = \int_0^h Cy(2h - y)\,dy = \frac{2}{3}Ch^3 = \frac{\boldsymbol{\rho g h^3 \sin\theta}}{\boldsymbol{3\mu}} \text{ per unit width} \quad \textit{Ans. (b)}$$