Some Definitions and Derivations for AMES 101A

Definition of a vector:

A three component object $\vec{\mathbf{v}}$ is a vector if its components (v_1,v_2,v_3) in coordinates $\mathbf{x} = (x_1,x_2,x_3)$ transform to components $(v_j l_{j1'}, v_j l_{j2'}, v_j l_{j3'})$ in new coordinates $\mathbf{x'}$, summing repeated indices, where $l_{ji'}$ is the cosine of the angle between the x_j axis and the x'_i axis. An example is the pressure gradient $p = (\partial p/\partial x_1, \partial p/\partial x_2, \partial p/\partial x_3)$. This is easily proved using the chain rule. If the coordinates are changed to $\mathbf{x'}$, then $\partial p/\partial x'_i = (\partial p/\partial x_j)(\partial x_j/\partial x_{i'})$, summing over j indices. But $\partial x_j/\partial x_{i'} = l_{ji'}$, so the pressure gradient satisfies the definition of a vector.

Definition of a second order tensor:

$\overleftrightarrow{\Pi}$ is defined to be a second order tensor if its components Π_{ij} transform according to the equation

$$\Pi_{i'j'} = \Pi_{mn}\, l_{mi'} l_{nj'}$$

Higher order tensors are defined in a similar way.

An obvious example is the inertial stress tensor $-\rho\mathbf{vv}$, with components $-\rho v_i v_j$. The components in $\mathbf{x'}$ coordinates are $-\rho v_{i'} v_{j'} = -\rho v_m l_{mi'} v_n l_{nj'}$, which satisfies the definition.

Leibnitz' rule for differentiating integrals:

$$\frac{d}{dt}\int_{CV} f\, dV = \int_{CV} \frac{\partial f}{\partial t}\, dV + \int_{CS} f\, \mathbf{v}_s \cdot d\mathbf{S}$$

where f is some function of space and time, CV is an enclosed control volume, CS is its surface with outward pointing surface vector $d\mathbf{S}$, and the velocity of the surface at $d\mathbf{S}$ is $\mathbf{v}_S$.

If $f = \rho\beta$, where β is the concentration of some conserved quantity per unit mass, then the total quantity B in a system (specified quantity of matter) is given by the integral over the control volume CV enclosing the system

$$B = \int_{CV} \rho\beta\, dV .$$

By Leibnitz' Rule, the rate of change of B with time for the system is found by setting the velocity of the surface $\mathbf{v}_S$ equal to the fluid velocity $\mathbf{v}$, since then, and only then, the control volume always encloses the system. For a general CV, we thus have the Reynolds transport theorem

$$\frac{dB}{dt} = \frac{d}{dt}\int_{CV} \rho\beta\, dV + \int_{CS} \rho\beta(\mathbf{v}-\mathbf{v}_S) \cdot d\mathbf{S} \text{ ; or } \frac{\partial B_{Syst.}}{\partial t} = \frac{\partial B_{CV}}{\partial t} + \int_{CS} \rho\beta\, \mathbf{v_r} \cdot d\mathbf{S}$$

where $\mathbf{v_r} = \mathbf{v} - \mathbf{v}_S$ is the relative velocity of the fluid with respect to the CV surface.

Derivation of energy conservation differential equations:

The first law of thermodynamics for a system is $\dot{Q} - \dot{W} = \dot{E}$, where $\dot{Q}$ is the rate of heat addition to the system through its surface by thermal conduction down a temperature gradient, $\dot{W}$ is the work being done on the surroundings by the system, and $\dot{E}$ is the rate of increase of the total energy of the system, including internal, kinetic, and potential energy. This "conservation of energy" law may be applied to a control volume surrounding a system of fluid, recognizing that the surface velocity of such a control volume moves with the velocity of the fluid **v**. For the system, $\dot{Q}$ is the integral over the system control surface in the first integral

$$\dot{Q} = \int_{CS} k \nabla T \cdot d\mathbf{S} = \int_{CV} \nabla \cdot (k \nabla T) dV$$

and the conversion is made to a volume integral using the divergence theorem in the second integral, on the right.

The work rate on the surroundings is composed of shaft work $\dot{W}_S$, the pressure work rate $\dot{W}_P$, and the viscous work rate $\dot{W}_V$. We will neglect the shaft work. The pressure and viscous work rates are found by integrating the corresponding surface forces on the surroundings, equal and opposite to the forces on the surface, dotted with the fluid velocity. The pressure force on the surface element $d\mathbf{S}$ is $-pd\mathbf{S}$, and the viscous force is $\tau \cdot d\mathbf{S}$, by Cauchy's rule. Therefore

$$\dot{W}_P = \int_{CS} p\mathbf{v} \cdot d\mathbf{S} = \int_{CV} \nabla \cdot (p\mathbf{v}) dV$$

and

$$\dot{W}_V = \int_{CS} -\mathbf{v} \cdot \overleftrightarrow{\tau} \cdot d\mathbf{S} = -\int_{CV} \nabla \cdot (\mathbf{v} \cdot \overleftrightarrow{\tau}) dV .$$

The rate of change of the total energy is given by

$$\dot{E} = \int_{CV} \frac{\partial(\rho e)}{\partial t} dV + \int_{CS} \rho e \mathbf{v} \cdot d\mathbf{S} = \int_{CV} \left[\frac{\partial(\rho e)}{\partial t} + \nabla \cdot (\rho e \mathbf{v}) \right] dV .$$

Collecting all terms in the energy equation into a single volume integral gives

$$\int_{CV} \left[\frac{\partial(\rho e)}{\partial t} + \nabla \cdot (\rho e \mathbf{v}) - \nabla \cdot (\mathbf{v} \cdot \overleftrightarrow{\tau}) + \nabla \cdot (p\mathbf{v}) - \nabla \cdot (k \nabla T) \right] dV = 0 .$$

The control volume was chosen arbitrarily and can be very small surrounding any point in the fluid. All the terms in the bracket become constant within the control volume when CV is small enough by the continuum hypothesis. Therefore bracketed quantities must be zero everywhere;

that is, identically zero, because the constant can be factored out of the integral giving [ ]CV=0. The differential equation expressing the conservation of energy is thus

$$\left[\frac{\partial(\rho e)}{\partial t} + \quad (\rho e\mathbf{v}) - \quad (\mathbf{v} \quad \overleftrightarrow{\tau}) + \quad (p\mathbf{v}) - \quad (k \quad T)\right] = 0 \; .$$

Expanding the first two terms and using the continuity equation and the definition of the substantive derivative (following the fluid particle) gives

$$\rho\frac{De}{Dt} - \quad (\mathbf{v} \quad \overleftrightarrow{\tau}) + \quad (p\mathbf{v}) - \quad (k \quad T) = 0 \; .$$

Conservation of momentum:

Newton's law for a fluid system applied to the same fluid system gives

$$\int_{CS} (-p\,\overleftrightarrow{\delta}) \; d\mathbf{S} + \int_{CS} \overleftrightarrow{\tau} \; d\mathbf{S} + \int_{CV} \rho\mathbf{g}\, dV = \int_{CV} \frac{\partial(\rho\mathbf{v})}{\partial t} dV + \int_{CS} \rho\mathbf{v}\mathbf{v} \; d\mathbf{S} \; .$$

Converting the surface integrals to volume integrals and collecting terms, we find

$$\int_{CV} \left[\frac{\partial(\rho\mathbf{v})}{\partial t} + \quad (\rho\mathbf{v}\mathbf{v}) - \quad \overleftrightarrow{\tau} + \quad p - \rho\mathbf{g}\right] dV = 0$$

where the bracketed term gives the differential equations of motion, expressing the conservation of linear momentum for a fluid

$$\frac{\partial(\rho\mathbf{v})}{\partial t} + \quad (\rho\mathbf{v}\mathbf{v}) - \quad \overleftrightarrow{\tau} + \quad p - \rho\mathbf{g} = 0 \; ,$$

which can also be written

$$\rho\frac{D\mathbf{v}}{Dt} - \quad \overleftrightarrow{\tau} + \quad p - \rho\mathbf{g} = 0 \; .$$

Taking the scalar product of this equation with **v** gives

$$\rho\frac{D(ke+pe)}{Dt} - \mathbf{v} \quad \overleftrightarrow{\tau} + \mathbf{v} \quad p = 0$$

which is the mechanical energy equation. Subtracting this from the first law energy equation above yields

$$\rho\frac{Du}{Dt} - \overleftrightarrow{\tau}: \ \mathbf{v} + p \quad \mathbf{v} - \quad (k \quad T) = 0$$

This shows that the internal energy of a fluid particle u changes by thermal conduction, compression work, and viscous dissipation, taking the last three terms from right to left.

Vorticity:

Define the vorticity field $\omega \equiv$ curl $\mathbf{v}$ = $\times\mathbf{v}$. Note that the i component of the cross product $\mathbf{C} = \mathbf{A}\times\mathbf{B}$ of two vectors $\mathbf{A}$ and $\mathbf{B}$ can be written

$$C_i = \varepsilon_{ijk}A_jB_k$$

where ε_{ijk} is the alternating tensor, defined such that $\varepsilon_{ijk} = 0$ if any two indices are equal, $\varepsilon_{ijk} = 1$ if ijk = 123, 231 or 312, and $\varepsilon_{ijk} = -1$ if ijk = 132, 321 or 132. Thus

$$\omega_i = \varepsilon_{ijk}\partial v_k/\partial x_j$$

is the i component of the vorticity vector. An important identity relating ε_{ijk} and the identity tensor δ_{ij}, where $\delta_{ij} = 0$ for $i \neq j$ and $\delta_{ij} = 1$ for i = j, is

$$\varepsilon_{ijk}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} .$$

Note that the curl of the gradient of any scalar field, such as a velocity potential φ or the pressure p, is zero. This is because curl grad φ = - curl grad φ as follows:

$$(\text{curl grad } \varphi)_i = \varepsilon_{ijk}(\partial^2\varphi/\partial x_j\partial x_k) = -\varepsilon_{ikj}(\partial^2\varphi/\partial x_j\partial x_k) = -\varepsilon_{ikj}(\partial^2\varphi/\partial x_k\partial x_j) = -\varepsilon_{ijk}(\partial^2\varphi/\partial x_j\partial x_k)$$

where the dummy indices jk are first reversed in ε_{ijk} to change the sign, then the order of the partial derivative $\partial^2\varphi/\partial x_j\partial x_k$ is reversed, and finally the indices are relabeled so $k\rightarrow j$ and $j\rightarrow k$.

Assuming ρ = constant, the equations of motion can be written

$$\partial v_i/\partial t + v_j\partial v_i/\partial x_j = -\partial(p/\rho)/\partial x_i + g_i + \nu \quad {}^2 v_i \quad .$$

Taking the curl of this equation, the first term gives the time derivative of the vorticity and the pressure term vanishes. The gravity term also drops out since it can be expressed as the gradient of the gravitational potential $-gx_3$. The nonlinear term $v_j\partial v_i/\partial x_j$ is rewritten as

$$v_j\partial v_i/\partial x_j = \partial(v^2/2)/\partial x_i - (\mathbf{v}\times\omega)_i$$

so the gradient term can be eliminated under the curl operation. This important vector identity is proved by expanding the "vortex force" term $(\mathbf{v}\times\omega)_i$ using $\varepsilon_{ijk}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$

$$(\mathbf{v}\times\omega)_i = \varepsilon_{ijk}v_j\varepsilon_{klm}\partial v_m/\partial x_l = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})v_j\partial v_m/\partial x_l = \partial(v^2/2)/\partial x_i - v_j\partial v_i/\partial x_j$$

and the index substitution property of δ_{jl}; that is, $v_i = v_j\delta_{ji}$ for example.

Taking the curl of the equations of motion then gives

$$\partial\omega_i/\partial t - \varepsilon_{ijk}\partial(\varepsilon_{klm}v_l\omega_m)/\partial x_l = \nu \quad {}^2\omega_i ,$$

which further reduces to

$$\partial\omega_i/\partial t + v_j\partial\omega_i/\partial x_j = \omega_j\partial v_i/\partial x_j + \nu\ \ {}^2\omega_i\ ,$$

the conservation of vorticity equation for constant density fluid. The left hand side is the substantive derivative $D\omega_i/Dt$, the first term on the right is the vorticity production by vortex line stretching, and the last term is the diffusion of vorticity by viscosity.

Gauss' Theorem:

Integrals may be converted by the expression (*Gauss' Theorem*)

$$\int_{CV} \frac{\partial A_{jkl...}}{\partial x_i}\, dV = \int_{CS} n_i A_{jkl...}\, dS$$

where $A_{jkl...}$ is a tensor and **n** is the outward pointing unit vector at elementary area dS of the control surface CS surrounding the control volume CV.

An important special case of Gauss' Theorem is the *divergence theorem*

$$\int_{CV} \frac{\partial A_i}{\partial x_i}\, dV = \int_{CS} n_i A_i\, dS$$

where **A** is a vector and its gradient tensor $\frac{\partial A_j}{\partial x_i}$ has been contracted to form the divergence $\frac{\partial A_i}{\partial x_i}$.

If **A** is the velocity **v**, then we can interpret the divergence of the velocity field **v** as the volume flow rate per unit volume $\dot{Q}/V$ emerging from a small control volume surrounding a point,

$$\int_{CV} \ \ \mathbf{v}\, dV = (\ \ \mathbf{v})\, V = \int_{CS} \mathbf{v}\ \ d\mathbf{S} = \dot{Q}\ ; \qquad \ \ \mathbf{v} = \dot{Q}/V$$

Therefore, **v** = 0 for an incompressible fluid.

The net force of a uniform pressure p_a on a control volume may be found from Gauss' theorem as follows,

$$\mathbf{F}_p = -\int p_a\mathbf{n}\ \ d\mathbf{S} = -\int \ \ p_a\, dV = 0$$

because the gradient of a constant is zero.

Derivation of Bernoulli's Equation:

Apply the conservation of mechanical energy equation

$$\rho\frac{D(ke+pe)}{Dt} - \mathbf{v} \qquad \overleftrightarrow{\tau} + \mathbf{v} \qquad p = 0$$

to the steady flow of fluid through a stream tube. Rewriting the equation

$$\rho\frac{\partial ke}{\partial t} = -\rho\mathbf{v} \quad ke - \mathbf{v} \quad p - \rho\mathbf{v} \quad gx_3 + \mathbf{v} \qquad \overleftrightarrow{\tau}$$

where ke is $v^2/2$. Assuming constant density and rearranging gives

$$\frac{\partial\rho ke}{\partial t} = - \qquad \rho\mathbf{v}(ke + p/\rho + gx_3) + \qquad (\mathbf{v} \quad \overleftrightarrow{\tau}) - \overleftrightarrow{\tau}:\frac{\partial\mathbf{v}}{\partial\mathbf{x}}$$

which can be used to evaluate the rate of change of the total kinetic energy KE in the stream tube

$$\frac{dKE}{dt} = \frac{d}{dt}\int_{CV}\rho\, ke\, dV = \int_{CV}\rho\frac{\partial ke}{\partial t}dV = \int_{CV}\left[- \qquad \rho\mathbf{v}(ke + p/\rho + gx_3) + \qquad (\mathbf{v} \quad \overleftrightarrow{\tau}) - \overleftrightarrow{\tau}:\frac{\partial\mathbf{v}}{\partial\mathbf{x}}\right]dV$$

$$= -\int_{CS}\rho\mathbf{v}(ke + p/\rho + gx_3)\ d\mathbf{S} + \int_{CS}\mathbf{v} \quad \overleftrightarrow{\tau} \quad d\mathbf{S} - \int_{CV}\left[\overleftrightarrow{\tau}:\frac{\partial\mathbf{v}}{\partial\mathbf{x}}\right]dV$$

$$= -\int_{CS}\rho\mathbf{v}B\ d\mathbf{S} - \dot{W}_V - \int_{CV}[\rho\varepsilon]\,dV$$

using the divergence theorem to convert to surface integrals and defining B and ε as indicated. For steady state flow through the stream tube the left hand side is zero. The first surface integral on the right is the convection rate into the stream tube of the Bernoulli energy/mass B, and equals the mass flow rate $\dot{m}$ times (B_1 - B_2). The second surface integral is the rate of viscous working on the streamtube -$\dot{W}_V$, and the third integral is the total rate of viscous dissipation within the volume. This can be written as

$$B_1 = B_2 + \frac{\dot{W}_V}{\dot{m}} + \frac{\rho V\bar{\varepsilon}}{\dot{m}} = B_2 + w_V + w_F$$

where the overbar indicates the volume average of the local viscous dissipation rate per unit mass ε, where

$$\varepsilon \equiv \frac{\overleftrightarrow{\tau}}{\rho}:\frac{\partial\mathbf{v}}{\partial\mathbf{x}}\ .$$

If viscous forces are negligible, we find the simplest form of the Bernoulli equation

$$B_1 = B_2\ ; \left(\frac{v^2}{2} + p/\rho + gx_3\right)_1 = \left(\frac{v^2}{2} + p/\rho + gx_3\right)_2 .$$

Extended forms of the Bernoulli equation take into account frictional losses, viscous working, and even time dependence.